



UNIVERSITÀ DEGLI STUDI DI MILANO
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Bachelor Degree in Physics

**Infrared-Safe NLO Calculations with Massive Quarks: An Extension of
the Nested Soft-Collinear Subtraction Formalism**

Supervisor:
Prof. Raoul Horst Röntsch

Student:
Leonardo Cerasi
Matr.: 11410A

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*« What we know is a drop,
what we don't know is an ocean.»*
I. Newton

Abstract

Precise predictions of hadronic scattering processes in particle colliders are gaining increasing importance in high-energy physics, as they both allow for testing the Standard Model (SM) and for probing potential signals of Beyond-SM physics (BSM). Given this growing experimental focus on precision measurements, especially at facilities like the Large Hadron Collider at CERN, there is a compelling need for precise theoretical estimates of observables sensitive to Quantum Chromodynamics (QCD) effects. Achieving greater precision requires the correction of leading-order (LO) calculations with (next-to-)leading (N^n LO) terms: these are generally challenging to compute, in part because they need to account for both real and virtual corrections to the considered hard processes, which both present infrared (IR) singularities stemming from unresolved partons (i.e. soft and/or collinear to other partons).

Although these individual corrections are divergent, the IR singularities cancel between them, ensuring that the total cross-section is finite even when including N^n LO corrections. One of the main difficulties of computing higher-order corrections is making the cancellation manifest. To do so, it is necessary to introduce subtraction methods for the regularization and extraction of these singularities in a general and local way on the unresolved phase space.

While a general solution has been found at NLO, most notably with the Catani–Seymour (CS) and the Frixione–Kunszt–Signer (FKS) subtractions schemes (SS), the problem remains open at NNLO: in the last two decades, a variety of schemes have been proposed, but none has reached the same level of generality as the CS and FKS schemes. One of the most promising ones is the Nested Soft-Collinear (NSC) SS, given its modularity and conceptual clarity: this SS regularizes IR divergences by factorizing soft and collinear singularities in a nested and sequential manner. This modular structure is particularly suited for an extension from NLO to NNLO, as well as for an implementation in Monte Carlo integration codes.

A limitation of the NSC SS is that it adopts the massless approximation for all quark flavours: this hinders the applicability of this SS as-is to processes involving heavy quarks (i.e. bottom and top quarks), which constitute one of the main areas of active research at particle colliders. The aim of this thesis is generalizing the NSC SS to consider completely generic partonic final states at NLO, with a general number of massless and massive quark flavours, in order to accommodate for the description of heavy-quark processes.

The analysis conducted in this thesis shows that the structure of the NSC SS is preserved when including massive final-state partons. Indeed, these partons only modify the soft and virtual operators $I_S(\epsilon)$ and $I_V(\epsilon)$, which factorize the divergences resulting from soft limits of real corrections and from virtual corrections respectively, while leaving the collinear operator $I_C(\epsilon)$ and the counterterms introduced by the collinear renormalization of PDFs unchanged. In particular, it is shown that ϵ^{-2} -poles in the sum $I_S(\epsilon) + I_V(\epsilon)$ cancel for general partonic processes, while ϵ^{-1} -poles are independent of massive partons and indeed cancel against the ϵ^{-1} -poles of $I_C(\epsilon)$, thus yielding an ϵ -finite reminder. Moreover, this finite reminder contains

massive logs of the kind $\sim \log m^2/Q^2$, with Q the hard energy scale, exactly as expected: this suggests that for studies at colliders like the LHC, with $Q \sim 100 \text{ GeV} - 1 \text{ TeV}$, the numerical stability of Monte Carlo codes restricts the quark flavours which can be considered massive to only the top and bottom flavours ($m_t \approx 172 \text{ GeV}$, $m_b \approx 4.18 \text{ GeV}$), as smaller masses can cause numerical instability due to large logarithms.

This thesis therefore lays the foundations for a future inclusion of general massive final states in NNLO QCD computations, thereby paving the way for precision studies of heavy-quark processes with the NSC SS.

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Introduction

The Standard Model of Particle Physics (SM) is, as of now, the most complete theoretical framework in subatomic physics, describing all known elementary particles and fundamental interactions [1–9], except for the very weak gravitational force. Over the last fifty years, the SM has been continuously tested via numerous experiments, mainly in the context of particle colliders, and its validity has been confirmed by the agreement of its predictions with experimental observations, culminating in 2012 with the discovery of the Higgs boson [10, 11] at the Large Hadron Collider (LHC) at CERN.

Despite its success, there is strong evidence for the existence of Physics Beyond the SM (BSM): the most prominent indications include the existence of dark matter and dark energy, the observed matter-antimatter asymmetry and the non-vanishing neutrino masses. Contrary to earlier expectations, though, since its first run in 2009 the LHC has not yet detected any new particle, nor any confirmation of BSM physics: instead, the huge amount of data collected in its three runs (Run 3 is currently ongoing) puts increasingly stricter exclusion limits on BSM models [12–15]. As a consequence, the allowed masses of hypothesized new particles have become so large that it is unlikely that they will be produced at the LHC in sufficient numbers for detailed studies.

Since a substantial further increase in the energy of colliding particles at the LHC (or any other collider) is currently not feasible, the lack of any observation of BSM physics at the LHC has sparked a change in the research paradigm in high-energy particle physics, as it is clear that BSM physics searches based on the idea of detectable resonant-like structures on top of flat backgrounds has to be supplemented by new research strategies. Indeed, new particles can still be produced at the LHC, though in a way which does not allow for their direct detection: undetected light particles could be hidden in complex final states, while heavy particles could be virtually produced for extremely short periods of time, before disappearing back into the quantum vacuum. In the latter case, these virtual particles could affect measurable properties, prompting their indirect detection as deviations from SM predictions.

Given this shift of focus towards higher experimental precision in collider physics, it is clear that reliable theoretical predictions of hadron-collision processes are needed.

§1.1 QCD in collider physics

In general, any precision study at a hadron collider is challenging to perform, given the poorly-understood nature of the strong force which keeps hadrons together. In fact, the strong interaction is described by Quantum Chromodynamics (QCD), which has the complicated mathe-

mathematical structure of a non-Abelian gauge theory (§2.1.1 for details).

Although it has not been possible, so far, to fully describe the properties of a single proton from first principles, in the context of hadron collisions a first-principles description is made possible for a particular class of processes: hard scattering processes, which are characterized by a large momentum transfer.

Even though hard scattering processes have a lower probability of happening, with respect e.g. to elastic scattering processes, they are of great interest to modern particle physics: in fact, these are the processes which produce electroweak (EW) bosons, the Higgs boson, new heavy particles, etc., thus allowing for their detailed study. Moreover, the analysis of hard scattering processes is facilitated by a remarkable property of QCD: **asymptotic freedom**.

Asymptotic freedom is a property of certain non-Abelian gauge theories. After the renormalization of a quantum field theory (QFT), the coupling constant of the interaction becomes dependent on a renormalization energy scale, a phenomenon known as the “running of the coupling” (see §2.2). The evolution of the running coupling $\alpha(\mu^2)$ as a function of the energy scale μ is described by the renormalization group equation (see e.g. Chapter 12 of [16]):

$$\mu^2 \frac{d\alpha(\mu^2)}{d\mu^2} = -2\beta(\alpha(\mu^2))\alpha(\mu^2) \quad (1.1)$$

where the Callan–Symanzik β -function has a power-series expansion like:

$$\beta(\alpha) = \sum_{n \in \mathbb{N}_0} \beta_n \left(\frac{\alpha}{4\pi} \right)^{n+1} = \beta_0 \frac{\alpha}{4\pi} + o(\alpha^2) \quad (1.2)$$

For QCD¹ $\beta_0 > 0$ (see [17, 18]), hence the coupling becomes small at high energies (small distances). This allows for a perturbative description of hard scattering processes: these events happen at small distances, hence the hadronic scattering can be studied through the interaction between individual partons (see Fig. 1.1), i.e. the quarks and gluons which compose the hadrons.

§1.1.1 Hadronic scattering

Given the asymptotic freedom of QCD, it is clear why hard scattering processes are preferred for precision studies at hadron colliders: these scatterings occur at high energy $Q \sim 100 \text{ GeV} - 1 \text{ TeV}$ (at the LHC), hence non-perturbative effects are suppressed by powers of Λ_{QCD}/Q , where $\Lambda_{\text{QCD}} \approx 200 \text{ MeV}$ is an energy scale describing the asymptotic freedom of QCD (see below). This is formalized by the factorization theorem [20], which states that hadronic cross-sections can be computed from partonic cross-sections as:

$$\begin{aligned} d\sigma_{h_1, h_2}(P_1, P_2) = \sum_{a, b} \int_{[0,1]^2} \frac{d\xi_1}{\xi_1} \frac{d\xi_2}{\xi_2} f_a^{(h_1)}(\xi_1, \mu_F^2) f_b^{(h_2)}(\xi_2, \mu_F^2) \times \\ \times d\hat{\sigma}_{a, b}(\xi_1 P_1, \xi_2 P_2, \alpha_s, \mu_R^2, \mu_F^2) \left[1 + o\left(\frac{\Lambda_{\text{QCD}}^n}{Q^n}\right) \right] \end{aligned} \quad (1.3)$$

with $n \in \mathbb{N}$. Here, the two scattering hadrons h_1, h_2 have momenta P_1, P_2 , while the scattering partons a, b have momentum fractions ξ_1, ξ_2 . For the rest of this work, the factorization scale

¹To be precise, for a $\text{SU}(n_c)$ gauge theory with $n_q < \frac{11}{2}C_A$ active quark flavours, see §2.1.2.

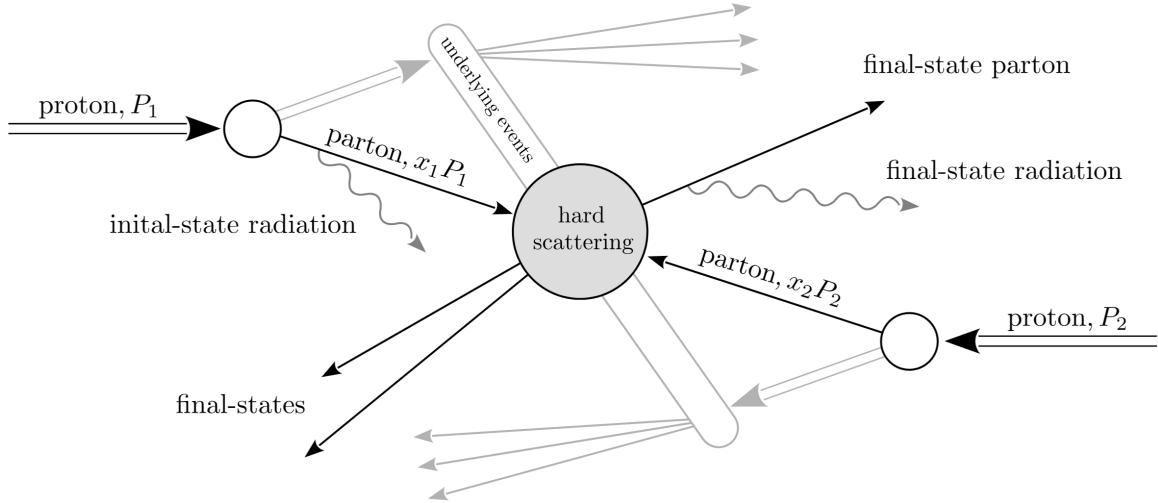


Figure 1.1: Schematics of hard hadronic scattering. Due to asymptotic freedom, individual partons can be assumed to be free particles, so that their (hard) scattering can be computed via perturbative QCD. Initial- and final-state radiation accounts for beyond-leading-order effects. Figure from [19].

μ_F is taken to be equal to the renormalization scale μ_R defined in §2.2.2. Finally, $d\hat{\sigma}_{a,b}$ is the partonic cross-section to be discussed in the next subsection.

The link between hadron-scale physics and parton-scale physics is given by **parton distribution functions** (PDFs): in general, $f_a^{(h)}(\xi)$ is the numerical probability of finding a parton a inside the hadron h with a definite momentum fraction ξ : $p_a = \xi P_h$, where p_a and P_h are the momenta of the parton and of the hadron, respectively. A crucial property of PDFs is their universality, as they are process-independent: this means that they can be measured in a particular process and then used in many others. However, they encapsulate non-perturbative effects which are poorly understood, thus they have not been computed from first principles so far.

Another instance of non-perturbative effects arises when considering that, after the partonic interaction, final-state partons can be clustered into so-called jets: despite the difficulty in formally defining jets (for a review of various jet algorithms, see [21]), they can be intuitively pictured as seeds of hadronic energy flows which are barely affected by non-perturbative QCD effects. However, while on short time-scales QCD can be treated perturbatively, on long time-scales QCD partons (and so jets too) are subject to the phenomenon of **hadronization**. Hadronization can be explained by considering a solution to Eq. 1.1 at leading-order in α , found by introducing a reference scale μ :

$$\alpha_s(\mu_R^2) = \frac{\alpha_s(\mu^2)}{1 + 2\alpha_s(\mu^2)\frac{\beta_0}{4\pi} \log \frac{\mu_R^2}{\mu^2}} \quad (1.4)$$

For example, $\alpha_s(m_Z^2) \approx 0.118$ [22]. It is customary to introduce a QCD scale Λ_{QCD} , defined as (see Chapter 2 of [23]):

$$\ln \frac{\mu_R^2}{\Lambda_{\text{QCD}}^2} \equiv \frac{1}{2} \int_{\alpha_s(\mu_R^2)}^{\infty} \frac{dx}{x\beta(x)} \quad (1.5)$$

Experimental analysis sets $\Lambda_{\text{QCD}} \approx 200$ MeV. Retaining only the β_0 term allows expressing the

running coupling as:

$$\alpha_s(\mu_R^2) \equiv \frac{1}{2 \frac{\beta_0}{4\pi} \log \frac{\mu_R^2}{\Lambda_{\text{QCD}}^2}} \quad (1.6)$$

This expression shows that $\mu_R \gg \Lambda_{\text{QCD}}$ is the perturbative region, where asymptotic freedom makes α_s small enough for perturbative techniques. On the other hand, for $\mu_R \rightarrow \Lambda_{\text{QCD}}$ a Landau pole is present: this pole signals the breakdown of perturbation theory and the hadronization of partons, i.e. their confinement into bound states (hadrons).

Eq. 1.3 can be represented graphically as in Fig. 1.2. Here, the hard scattering process occurs at high energy $Q \gg \Lambda_{\text{QCD}}$, resulting in jets which are initially unaffected by non-perturbative QCD, since their energy is well above the QCD scale; however, this energy is radiated off in the form of parton showers, and, when the threshold energy Λ_{QCD} is reached, non-perturbative effects come into play, resulting in the hadronization of jets.

The factorization theorem also provides a numerical estimate for corrections due to non-perturbative effects. Setting the lowest energy scale of the considered scattering (e.g. a p_T -cut on a jet) to be $Q \approx 20 \text{ GeV}$ and $n = 1$ in Eq. 1.3, then non-perturbative corrections are of order $\sim 10^{-2}$, i.e. percentage-level corrections.

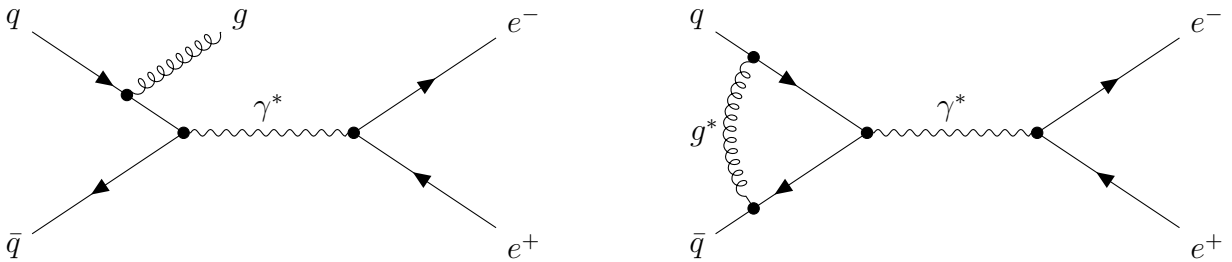
§1.1.2 Partonic scattering

For the rest of this work, the analysis is restricted to partonic scattering treated using perturbation theory. Thus, the partonic cross-section for the scattering of two partons a, b with momenta p_1, p_2 can be expressed as a power series in the running coupling:

$$d\hat{\sigma}_{a,b}(p_1, p_2) = \sum_{n \in \mathbb{N}_0} d\hat{\sigma}_{a,b}^{(n)}(p_1, p_2) \quad (1.7)$$

where each term is $d\hat{\sigma}^{(n)} \sim \alpha_s^{n_0+n}$, with $n_0 \in \mathbb{N}$ giving the dependence on $\alpha_s(\mu_R^2)$ due to the leading-order (LO) process, which is usually (but not always) a tree-level process.

The $n \geq 1$ terms are referred to as QCD corrections. Focusing on next-to-leading-order (NLO) corrections, they can be of two kinds: real corrections and virtual corrections. Real corrections consist in the emission of an additional parton as initial- or final-state radiation, while virtual corrections present an additional partonic loop. Examples of a real and a virtual correction to the Drell-Yan process may be:



In general, then:

$$d\hat{\sigma}_{a,b}^{(1)}(p_1, p_2) = d\hat{\sigma}_{a,b}^{\text{R}}(p_1, p_2) + d\hat{\sigma}_{a,b}^{\text{V}}(p_1, p_2) + d\hat{\sigma}_{a,b}^{\text{pdf}}(p_1, p_2) \quad (1.8)$$

where $d\hat{\sigma}_{a,b}^{\text{R}}$ and $d\hat{\sigma}_{a,b}^{\text{V}}$ are the single-real and one-loop corrections. The additional correction $d\hat{\sigma}_{a,b}^{\text{pdf}}$ is due to the collinear renormalization of PDFs.

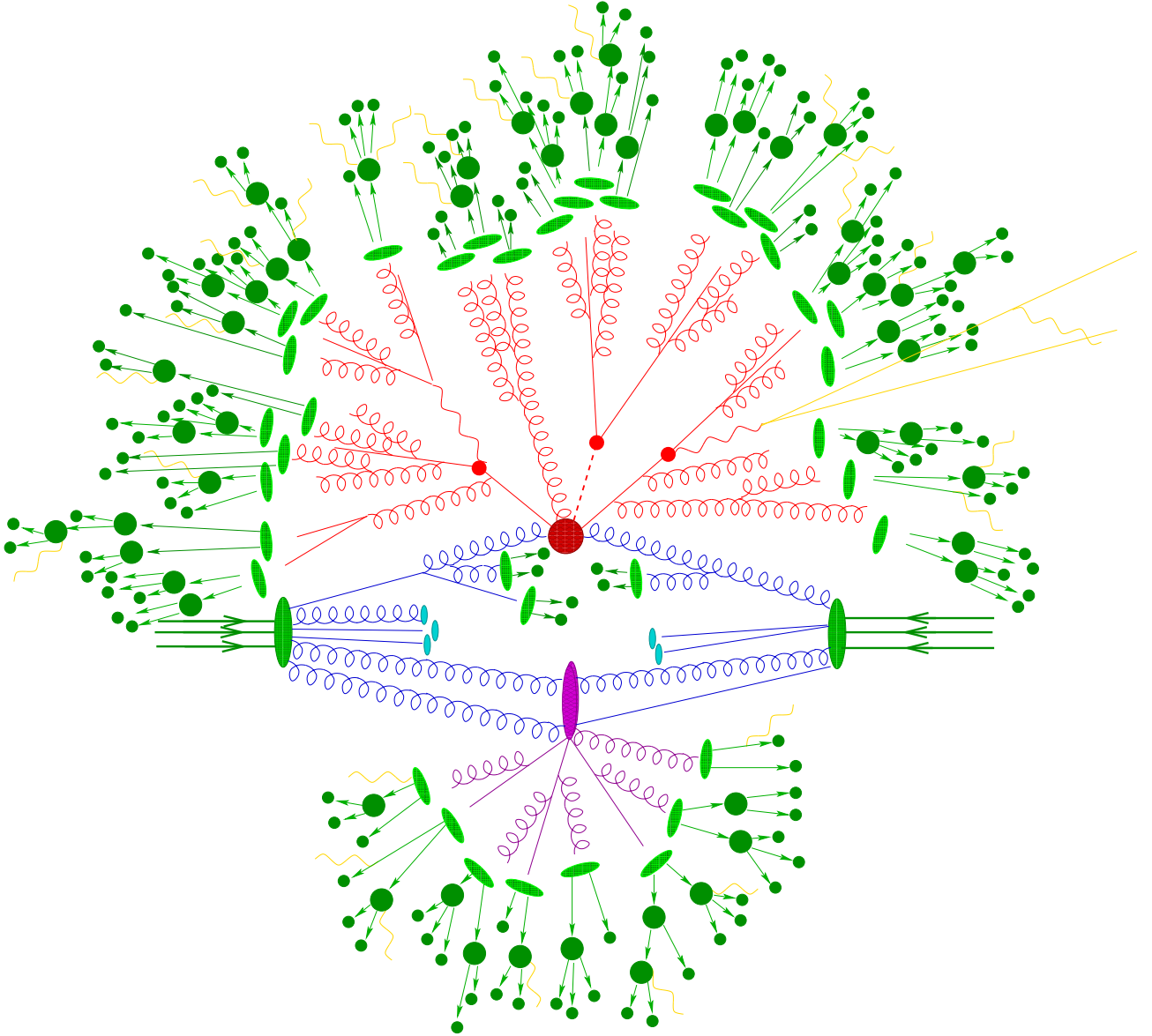


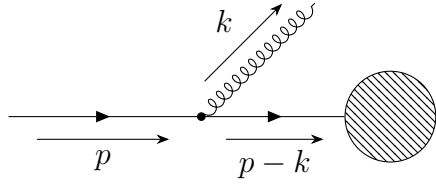
Figure 1.2: Hadronization of jets produced in a hard hadronic scattering. Incoming hadrons produce incoming partons (blue) which, after emitting some initial-state radiation, enter the hard scattering event (red blobs) as well as a double parton scattering (purple blob, not considered in this work): these scatterings give rise to partonic jets (red and purple) which undergo hadronization (light green blobs), eventually forming heavy hadrons (dark green blobs) and producing soft radiation (yellow). Figure from [24].

§1.2 Singularities in QCD amplitudes

One of the main difficulties when computing real and virtual corrections to scattering amplitudes is the presence of singularities in particular kinematic limits.

§1.2.1 Infrared poles

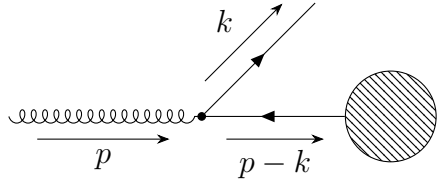
In the case of amplitudes with real emissions, singularities arise when the energy of a gluon vanishes (**soft singularity**) or when two massless partons are emitted in the same direction (**collinear singularity**). To illustrate why the amplitude diverges in these limits, consider a real-emission diagram like:



$$\sim \frac{1}{(p-k)^2} = -\frac{1}{2E_p E_k (1 - \cos \theta)}$$

where the massless approximation² for the quark is employed. It is then clear that the amplitude diverges for $E_k \rightarrow 0$ and $\theta \rightarrow 0$ due to the propagator of the virtual quark; note that single quark emissions do not give rise to any soft singularities, as in the massless limit they provide an integrable singularity (due to the presence in the matrix element of spinors $u^s(p) \sim \sqrt{E_p}$). Both kinds of singularities in real emissions can be seen as one virtual massless parton going on-shell (in the above example, the virtual quark with momentum $q = p - k$ in the massless approximation), i.e. with $q^2 \rightarrow 0$, and they are collectively called infrared-collinear singularities, or commonly (though less precisely) infrared (IR) singularities.

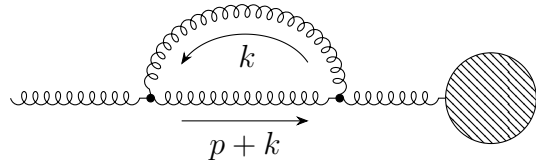
It is important to note that massive partons do not determine soft or collinear singularities, as shown e.g. by $g \rightarrow q + \bar{q}$ splitting:



$$\sim \frac{1}{(p-k)^2 - m^2} = -\frac{1}{2E_p (E_k - |\mathbf{k}| \cos \theta)}$$

Obviously there is no soft limit in this case, as $E_k \geq m > 0$. Moreover, as $E_k \neq |\mathbf{k}|$ for $m > 0$, the collinear limit $\theta \rightarrow 0$ does not produce a singularity.

The case of virtual corrections is more complex. Indeed, virtual-correction amplitudes present an additional loop integral, which has two kinds of singularities: ultraviolet (UV) singularities and IR ones. To illustrate them, consider the following loop diagram:



$$\sim \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2} \frac{1}{(p+k)^2}$$

²In the context of collider physics, light quarks are usually approximated as massless, as they have $m_q/Q \lesssim 10^{-3}$, with $Q \sim 100 \text{ GeV} - 1 \text{ TeV}$ in a typical LHC process. The only exceptions are the heavy flavours: the bottom quark is sometimes treated as massive ($m_b \approx 4.18 \text{ GeV}$), while the top quark is always treated as massive ($m_t \approx 172 \text{ GeV}$). Data from [22].

The UV divergence can be seen performing a Wick rotation $k^0 \mapsto ik^0$ and introducing a cutoff Λ for the Euclidean momentum's magnitude $|k_E|$: in the UV limit $k^2 \gg p^2$, so the integral is $\sim \log \Lambda$, which is clearly divergent for $\Lambda \rightarrow \infty$. However, this kind of divergences is cured through the procedure of renormalization (§2.2).

IR divergences, on the other hand, have the same origin as those in real corrections: in reality, Nature does not distinguish between “real” and “virtual” corrections, which are merely human-made categories introduced to simplify the calculations. To confirm this, while real and virtual corrections present IR divergences when considered individually, these poles have to cancel when the two sets of corrections are added together: this is an instance of the Kinoshita–Lee–Nauenberg theorem, which asserts that the SM is IR-finite [25, 26].

§1.2.2 Subtraction method

The central problem of showing the pole cancellation is the fact that real and virtual corrections have different particle multiplicity in the phase space, so their summation is not straightforward. However, Catani's formula [27] allows one to extract IR singularities from renormalized virtual-correction amplitudes as poles in the number of spacetime dimensions (see §2.2.1). Luckily, the same can be done with real corrections too: in the IR limits a parton goes unresolved, so the effective phase space has the same multiplicity as that of virtual corrections (see §3.1).

The crucial point of extracting divergences, and in particular from real corrections, is to do so without integrating over the resolved phase space, in order to obtain an expression for the fully-differential (with respect to resolved partons) cross-section which allows for the numerical evaluation of phase-space integrals for any IR-safe observable. To do so, it is necessary to introduce a subtraction method.

To illustrate the idea of a subtraction method, consider the following integral:

$$I = \int_0^1 \frac{dx}{x^{1+\epsilon}} f(x) \quad (1.9)$$

where $f(x)$ is an arbitrary integrable function regular at $x = 0$. The integrand clearly diverges at $x = 0$, and this singularity is regulated by the parameter ϵ which leads to a $\frac{1}{\epsilon}$ pole after integration. The aim of the subtraction method is to analytically extract this pole and regulate the integral, making it finite in the $\epsilon \rightarrow 0$ limit. The basic idea is to introduce a counterterm by writing $f(x) = [f(x) - f(0)] + f(0)$, so that:

$$I = \int_0^1 \frac{dx}{x^{1+\epsilon}} [f(x) - f(0)] + f(0) \int_0^1 \frac{dx}{x^{1+\epsilon}}$$

The second integral is analytically determined for $\Re \epsilon < 0$, hence:

$$I = \int_0^1 \frac{dx}{x^{1+\epsilon}} [f(x) - f(0)] - \frac{f(0)}{\epsilon} \quad (1.10)$$

The pole has successfully been isolated, and the integral is now regular at $x = 0$, allowing for its numerical integration after setting $\epsilon \rightarrow 0$.

In a similar fashion, subtraction schemes allow for the extraction of IR singularities from QCD corrections. The main challenge is to define $f(0)$ such that it locally removes all divergences in the first term of Eq. 1.10, while also allowing the analytic integration of the second term, in

a process-independent way. At NLO this problem has been completely solved in a fully-local and analytic way with the Catani–Seymour (CS) scheme [28] and the Frixione–Kunszt–Signer (FKS) scheme [29, 30]. At NNLO a completely general, analytic and fully-local subtraction scheme has not yet been developed, but there are several candidates: in this work we consider the Nested Soft-Collinear (NSC) subtraction scheme (SS), introduced in [31].

The NSC SS can be applied to any scattering process with a generic number of final-state massless partons [32]; however, it cannot yet describe processes with final-state massive partons, e.g. top quarks. This work takes the first step in this direction, generalizing the NLO subtraction operators defined in [32–34] to account for both massless and massive final-state partons. We first give a general discussion on QCD in chapter 2, focusing on the renormalization scheme used and the colour-space formalism for the amplitudes. Then, we provide an overview of the NSC SS in chapter 3, before extending it to include massive final-state partons in chapter 4. Finally, we conclude in chapter 5 with possible future developments.

Preliminaries

§2.1 Quantum Chromodynamics

We consider a generalized QCD with gauge group $SU(n_c)$, with n_c colours and $n = n_f + n_F$ total quark flavours (n_f massless and n_F massive quark flavours). Note that in the SM $n_c = 3$ and $n = 6$.

§2.1.1 Yang–Mills theories

A quantum field theory can be built starting from its symmetry properties: in particular, specifying a group of local transformations, the **gauge group**, under which the theory must be invariant. Historically, the idea of gauge theories was first explored by Yang and Mills in [35], with the aim of studying isotopic gauge invariance for the nucleon, and then generalized by Utiyama in [36]. A modern treatment of gauge theories can be found in Chapter 15 of [16], which we follow for our discussion.

Consider n fermionic fields $\{\psi_k(x)\}_{k=1,\dots,n}$ and an n -spinor $\Psi(x)$ defined as:

$$\Psi(x) = \begin{pmatrix} \psi_1(x) \\ \vdots \\ \psi_n(x) \end{pmatrix} \quad (2.1)$$

As a gauge group, consider a d -dimensional Lie group G : in particular, take G to be simply-connected, so that each element can be expressed via the exponential map, and compact, so that its representations are unitary. Then, consider $\{T^a\}_{a=1,\dots,d} \subset \mathbb{C}^{n \times n}$ a representation of the associated Lie algebra \mathfrak{g} , so that the action of G on Ψ can be expressed as:

$$\Psi(x) \mapsto V(x)\Psi(x) \quad V(x) := \exp[i\theta_a(x)T^a] \quad (2.2)$$

where the Lie parameters $\{\theta_a(x)\}_{a=1,\dots,d} \subset \mathcal{C}^\infty(\mathbb{R}^{1,3})$ define a local gauge transformation. The aim is to define a Lagrangian which is invariant under this transformation, i.e. the Lagrangian of a (local) gauge theory.

Simple terms invariant under global phase rotations, like the fermion mass term $m\bar{\Psi}\Psi$, are of course invariant under Eq. 2.2 too, but derivatives need a careful treatment: indeed, the limit-definition of a derivative involves fields at different spacetime points, which have different transformations according to Eq. 2.2. In order to define a derivative of Ψ , it is necessary to

introduce a factor to subtract values of $\Psi(x)$ in a meaningful way, so consider $U(y, x) \in U(n) : U(x, x) = 1$ and which transforms under the action of G as:

$$U(y, x) \mapsto V(y)U(y, x)V^\dagger(x) \quad (2.3)$$

By the unitarity of the representations of G , it is clear that $U(y, x)\Psi(x)$ and $\Psi(y)$ have the same transformation law, so they can be meaningfully subtracted. Then, given $n^\mu \in \mathbb{R}^{1,3}$, the covariant derivative of a fermionic field $\Psi(x)$ along n^μ is defined as:

$$n^\mu D_\mu \Psi(x) := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} [\Psi(x + \varepsilon n) - U(x + \varepsilon n, x)\Psi(x)] \quad (2.4)$$

where $U(y, x)$ is defined through Eq. 2.3. To make this definition explicit, it is necessary to get an expression of $U(y, x)$ at infinitesimally-separated points. Given the unitarity of $U(y, x)$, it can be expressed through the generators $\{T^a\}_{a=1, \dots, d}$ as:

$$U(x + \varepsilon n, x) = I_n + ig\varepsilon n^\mu A_\mu^a(x)T_a + o(\varepsilon^2) \quad (2.5)$$

where $g \in \mathbb{R}$ is a constant. The new vector field $A_\mu^a(x)$ (actually, d different vector fields) is a **connection**, and it allows us to express the covariant derivative as (directly from Eq. 2.4):

$$D_\mu = \partial_\mu - igA_\mu^a T_a \quad (2.6)$$

To show that $D_\mu \Psi$ transforms in the same way as Ψ , note that, from Eq. 2.3-2.5:

$$I_n + ig\varepsilon n^\mu A_\mu^a(x)T_a \mapsto I_n - \varepsilon n^\mu V(x)\partial_\mu V^\dagger(x) + V(x)(ig\varepsilon n^\mu A_\mu^a(x)T_a)V^\dagger(x) + o(\varepsilon^2)$$

Hence, the connection transforms as:

$$A_\mu^a(x)T_a \mapsto V(x) \left[A_\mu^a(x)T_a + \frac{i}{g}\partial_\mu \right] V^\dagger(x) = A_\mu^a(x)T_a - f^{abc}A_\mu^a(x)\theta^b(x)T_c + \frac{1}{g}\partial_\mu \theta^a(x)T_a + o(\theta^2)$$

The second term makes it clear that the connection transforms according to the adjoint representation. From this expression, it follows that:

$$D_\mu \Psi(x) \mapsto [I_n + i\theta^a(x)T_a + o(\theta^2)] (\partial_\mu - igA_\mu^a(x)T_a) \Psi(x) = V(x)D_\mu \Psi(x)$$

where the relation $T_a T_b - if^{abc}T_c = T_b T_a$ of the associated Lie algebra was used.

The gauge-invariant Lagrangian can thus be built using covariant derivatives (minimal coupling prescription), but one also needs to include a kinetic term for the connection, i.e. a gauge-invariant term dependent on $A_\mu^a(x)$ only. This term can be found considering the commutator of covariant derivatives:

$$[D_\mu, D_\nu] = -igF_{\mu\nu}^a T_a \quad (2.7)$$

with the **field-strength tensor** defined as:

$$F_{\mu\nu}^a := \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c \quad (2.8)$$

Note that the field-strength tensor is not itself a gauge-invariant quantity, as really there are d different field-strength tensors; however, it is straightforward to construct gauge-invariant combinations of $F_{\mu\nu}^a$. In fact, in general any globally-symmetric function of Ψ , $F_{\mu\nu}^a$ and their

covariant derivatives is also locally-symmetric, i.e. gauge-invariant: this follows from the construction of the covariant derivative. For a complete discussion, see Chapter 15 of [16]. Usually, the following gauge-invariant term is taken as kinetic term for the gauge field (i.e. the connection $A_\mu^a(x)$):

$$\text{tr}\{(F_{\mu\nu}^a T_a)^2\} = 2F_{\mu\nu}^a F_a^{\mu\nu} \quad (2.9)$$

This allows defining the simplest non-Abelian gauge theory, **Yang–Mills theory**:

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4}F_{\mu\nu}^a F_a^{\mu\nu} \quad (2.10)$$

To account for fermions interacting with the gauge field $A_\mu^a(x)$, the Dirac Lagrangian with minimal coupling is added (see Chapter 15 of [37]):

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^a F_a^{\mu\nu} + \bar{\Psi} (i\not{D} - m) \Psi \quad (2.11)$$

§2.1.2 Gauge group $\text{SU}(n_c)$

The $\text{SU}(n_c)$ group is the group of unitary transformations of n_c -dimensional complex vectors. Its (faithful) fundamental representation thus is:

$$\text{SU}(n_c) = \{U \in \mathbb{C}^{n_c \times n_c} : UU^\dagger = U^\dagger U = I_{n_c} \wedge \det U = +1\}$$

The generators of $\text{SU}(n_c)$ can be found setting $U = \exp(i\theta_a T^a) = I_{n_c} + i\theta_a T^a + o(\theta^2)$ and using $U^\dagger U = I_{n_c}$:

$$T^a = T^{a\dagger} \quad (2.12)$$

Moreover, by the Jacobi formula $(\det A(t)) \frac{d}{dt} (\det A(t)) = \text{tr}(A(t)^{-1} \frac{d}{dt} A(t))$ evaluated at $t = 0$:

$$\text{tr} T^a = 0 \quad (2.13)$$

The traceless condition can be generalized to all semi-simple Lie algebras. Therefore, the generators of $\text{SU}(n_c)$ are $\mathbb{C}^{n_c \times n_c}$ Hermitian traceless matrices: the dimension of $\mathfrak{su}(n_c)$ then is $n_c^2 - 1$.

In general, the adjoint representation of a Lie group is given by representing its generators (i.e. the basis of the Lie algebra) with the structure constants of the Lie algebra:

$$(T_{\text{ad}}^b)_{ac} \equiv \bar{T}_{ac}^b = i f^{abc} \quad (2.14)$$

which, in the case of $\text{SU}(n_c)$, are $f^{abc} = \epsilon^{abc}$. Indeed, it can be shown by the Jacobi identity that the structure constants satisfy the Lie algebra:

$$f^{abd} f^{dce} - f^{acd} f^{dbe} = f^{bcd} f^{ade} \iff [[T^a, T^b], T^c] + [[T^c, T^a], T^b] + [[T^b, T^c], T^a] = 0$$

Moreover, since the structure constant are real, the adjoint representation is always a real representation: the adjoint representation of $\text{SU}(n_c)$ has degree $n_c^2 - 1$.

Representation are labelled by their Casimir operators. For any simple Lie algebra, given a representation \mathbf{r} , a Casimir operator is defined as:

$$T_{\mathbf{r}}^a T_{\mathbf{r}}^a = C_2(\mathbf{r}) I_{n_{\mathbf{r}}} \quad (2.15)$$

This is called the **quadratic Casimir operator**, as it is associated to $T^2 \equiv T^a T^a$ (a Casimir operator since $[T^b, T^2] = i f^{bac} \{T^c, T^a\} = 0$ by antisymmetry). For the fundamental and the adjoint representations \mathbf{n} and \mathbf{g} of $\text{SU}(n_c)$, the quadratic Casimir operators are (§A.3):

$$C_F \equiv C_2(\mathbf{n}) = T_R \frac{n_c^2 - 1}{n_c} \quad C_A \equiv C_2(\mathbf{g}) = 2T_R n_c \quad (2.16)$$

where T_R (usually taken to be $T_R = \frac{1}{2}$) is the trace normalization of the generators in the fundamental representation:

$$\text{tr}(T_{\mathbf{n}}^a T_{\mathbf{n}}^b) = T_R \delta^{ab} \quad (2.17)$$

§2.1.3 Quantization

The quantization of a non-Abelian gauge theory requires careful treatment due to the presence of spurious non-physical degrees of freedom. These can be accounted for with the Faddeev–Popov method [38], i.e. imposing a gauge fixing condition and introducing non-physical ghost fields, which serve as “negative” degrees of freedom. Then, functional quantization can be performed naturally, resulting in the following Feynman rules for the Yang–Mills Lagrangian (excluding ghost fields):

$$\begin{aligned}
 a \xrightarrow{p} b &= \frac{i}{\not{p} - m} \delta_{ab} & \mu, a \xrightarrow{k} \nu, b &= \frac{-i g_{\mu\nu}}{k^2} \delta_{ab} \\
 \begin{array}{c} \mu, a \\ | \\ \bullet \\ / \quad \backslash \\ b \quad c \end{array} &= i g \gamma^\mu T^a \delta_{bc} & \begin{array}{c} \mu, a \\ | \\ \bullet \\ / \quad \backslash \\ \nu, b \quad \rho, c \end{array} &= g f^{abc} [g^{\mu\nu} (p - q)^\rho + g^{\nu\rho} (q - k)^\mu + g^{\rho\mu} (k - p)^\nu] \\
 \begin{array}{c} \mu, a \quad \nu, b \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ \rho, c \quad \sigma, d \end{array} &= -i g^2 [f^{abe} f^{cde} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) + f^{ace} f^{bde} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho}) + f^{ade} f^{bce} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma})]
 \end{aligned}$$

The peculiar nature of non-Abelian gauge theories is due to the possibility for gauge bosons to self-interact (last two vertices).

In the case of QCD, the quanta of the n spinor fields are the n quark flavours, while the $n_c^2 - 1$ gauge fields are gluon fields, and their colour charges are given by their respective $\text{SU}(n_c)$ representations: n_c charges for the quarks (fundamental n -dimensional representation) and $n_c^2 - 1$ charges for the gluons (adjoint representation).

§2.2 Renormalization scheme

The computation of NLO corrections to scattering processes in a generic QFT involves UV-divergent loop amplitudes. In order to obtain finite results from these divergences, a renormalization scheme must be implemented.

The generalization of Catani's formula for IR singularities in virtual corrections is provided in [39] in a charge-unrenormalized (but mass-renormalized) way, i.e. still containing UV singularities (recall §1.2), thus it is necessary to carry out the renormalization procedure explicitly. To this end, we formally state the renormalization scheme adopted in this work.

§2.2.1 Dimensional regularization

As stated above, in the evaluation of loop amplitudes, both UV- and IR-singularities are encountered. The most efficient way to simultaneously regularize both types of divergences is dimensional regularization [40, 41].

In general, the dimensional regularization scheme consists in the analytic continuation of loop momenta to $d = 4 - 2\epsilon$ dimensions, with $\epsilon \in \mathbb{C} : \Re \epsilon < 0$ for IR divergences and $\Re \epsilon > 0$ for UV ones. This procedure turns loop integrals into meromorphic functions of $\epsilon \in \mathbb{C}$, allowing for the isolation of divergences as poles in ϵ .

The dimensional regularization prescription leaves freedom in choosing the dimensionality of external momenta, as well as the number of polarizations of both external and internal particles, thus allowing for the definition of different regularization schemes. We choose to work with **conventional dimensional regularization** (CDR), in which all momenta and polarization are analytically continued to d dimensions, as opposed to the 't Hooft–Veltman scheme (HV), in which only internal momenta and polarizations are.

When considering non-chiral gauge theories like QCD, CDR is the most natural choice, as the main difference between CDR and HV is the treatment of purely 4-dimensional objects, i.e. γ^5 and $\epsilon_{\mu\nu\sigma\rho}$. In particular, in CDR both the Dirac algebra and Lorentz indices are analytically continued to d dimensions, leading to a mathematical inconsistency stemming from the fact that, when $d \notin \mathbb{N}$, the following identities cannot hold simultaneously¹:

$$\{\gamma^5, \gamma^\mu\} = 0 \quad \forall \mu = 0, 1, \dots, d-1 \quad \text{tr}\{\gamma^5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma\} = -4i\epsilon^{\mu\nu\rho\sigma}$$

The choice of CDR over HV is then clear: in QCD, the only pathological objects are encountered when considering chiral vertices (e.g. for pseudoscalar mesons) and electroweak interactions, and both can be handled via known prescriptions, e.g. the Breitenlohner-Maison/'t Hooft-Veltman (BMHV) scheme [42] or the Larin scheme [43].

§2.2.2 Minimal subtraction

Once regularized, UV-divergences have to be removed via renormalization of fields and coupling constants. As a result of the renormalization procedure, a running coupling $\alpha_s(\mu_R^2)$ is introduced, and its definition in terms of the bare coupling $\alpha_{s,b}$ depends both on the regularization and the renormalization schemes.

In this work, we renormalize the coupling in a standard way (as in [27]) using the **modified minimal-subtraction scheme** ($\overline{\text{MS}}$), which directly subtracts UV-divergences from the coupling:

$$\alpha_{s,b} S_\epsilon = \alpha_s(\mu_R^2) \mu_R^{2\epsilon} \left[1 - \frac{1}{2} \frac{\alpha_s(\mu_R^2)}{2\pi} \frac{\beta_0}{\epsilon} + o(\alpha_s^2) \right] \quad (2.18)$$

¹This inconsistency is the explicit manifestation of a more profound topological issue of analytically continuing the number of dimensions: the Levi-Civita symbol in $d = 4$ is linked to the Grassmann algebra $\bigwedge(\mathbb{R}^{1,3})$, and in particular to its top-form, but $\bigwedge^k(\mathbb{R}^d)$ is only defined for $d \in \mathbb{N}$, so the top exterior subspace $\bigwedge^d(\mathbb{R}^{1,d-1})$ is meaningless for $d \notin \mathbb{N}$ and the Levi-Civita symbol cannot be analytically continued to $d = 4 - 2\epsilon$ dimensions.

where μ_R is an arbitrary renormalization scale, S_ϵ is the typical phase-space volume factor in dimensional regularization:

$$S_\epsilon \equiv (4\pi)^\epsilon e^{-\gamma_E \epsilon} \quad (2.19)$$

with $\gamma_E = 0.5772\dots$ the Euler-Mascheroni constant, and β_0 is the leading-order coefficient of the QCD β -function Eq. 1.2:

$$\beta_0 := \frac{11}{6}C_A - \frac{2}{3}T_R n_q \quad (2.20)$$

where n_q is the number of active quark flavours at the considered energy scale. In this work $n_q = n_f$ unless otherwise specified.

An important clarification about the dimensionality of $\alpha_{s,b}$ and α_s is needed, due to the presence of $\mu_R^{2\epsilon}$ in Eq. 2.18. In dimensional regularization, the action remains a dimensionless quantity², hence, given $\mathcal{S} = \int d^d x \mathcal{L}$, the QCD Lagrangian Eq. 2.11 must have dimension $[\mathcal{L}] = d$, as $[d^d x] = -d$. It is now trivial to verify the following dimensions:

$$[\Psi] = \frac{d-1}{2} \quad [A_\mu^a] = \frac{d-2}{2} \quad [g] = \frac{4-d}{2} = \epsilon$$

This shows that, in dimensional regularization, $[\alpha_{s,b}] = 2\epsilon$. In order to work with dimensionless quantities, then, in Eq. 2.18 we chose to extract the mass dimension from α_s .

When dealing with scattering processes, a fundamental quantity is the amplitude of a process. In the Schrödinger picture, given a quantum system described by a Hamiltonian H and a Hilbert space \mathcal{H} , the amplitude for the process $|a\rangle \rightarrow |b\rangle$, where $|a\rangle, |b\rangle \in \mathcal{H}$ is defined as:

$$\mathcal{A} := \langle b | S(t, t_0) | a \rangle \quad (2.21)$$

where $\Delta t \equiv t - t_0$ is the time elapsed in the transition and the S -matrix is defined as:

$$S(t, t_0) := e^{-iH(t-t_0)} \quad (2.22)$$

For an $n \rightarrow m$ scattering with defined initial- and final-state momenta, the amplitude can be written in the multi-particle phase space as:

$$\mathcal{A}_{n+m} := \langle \mathbf{p}_1, \dots, \mathbf{p}_n | S(+\infty, -\infty) | \mathbf{k}_1, \dots, \mathbf{k}_m \rangle \quad (2.23)$$

where $t, t_0 \rightarrow \pm\infty$ as to consider free-particle initial and final states (for a discussion on this adiabatic approximation, see Chapter 5 of [44]). The explicit expression of the amplitude can be computed from Feynman diagrams: for a complete discussion, see Chapter 4 of [16].

In general, we consider amplitudes \mathcal{A}_m involving m external QCD partons (gluons and quarks), with momenta $\{p\} \equiv \{p_1, \dots, p_m\}$, and an arbitrary number of colourless particles (photons, leptons, ...). Dependence on the momenta and quantum numbers of colourless particles is always understood and not explicitly shown in this work. The $\overline{\text{MS}}$ -renormalized amplitude has the following perturbative expansion in α_s :

$$\mathcal{A}_m(\alpha_s(\mu_R^2), \mu_R^2; \{p\}) = \left(\frac{\alpha_s(\mu_R^2)}{2\pi} \right)^q \left[\mathcal{A}_m^{(0)}(\mu_R^2; \{p\}) + \frac{\alpha_s(\mu_R^2)}{2\pi} \mathcal{A}_m^{(1)}(\mu_R^2; \{p\}) + o(\alpha_s^2) \right] \quad (2.24)$$

where the overall power is, in general, $q \in \frac{1}{2}\mathbb{N}_0$. Note that, although UV-divergences have been removed by the renormalization procedure, these amplitudes are still IR-singular as $\epsilon \rightarrow 0$.

²Recall that, in natural units $c = \hbar = 1$, all dimensions can be expressed as mass dimensions, since $[T] = [L] = [M]^{-1}$.

§2.3 Colour-space formalism

To handle the colour structure of QCD amplitudes, we adopt the colour-space formalism as in [28].

The m external partons in the amplitude \mathcal{A}_m each carry two indices: a colour index and a spin index. Colour indices are denoted by c_1, \dots, c_m : for gluons $c_i \equiv a_i \in \{1, \dots, n_c^2 - 1\}$, as the field-strength tensor Eq. 2.8 transforms according to the adjoint representation of the gauge group, while for quarks $c_i \equiv \alpha_i \in \{1, \dots, n_c\}$, as their Dirac fields transform according to the fundamental representation of the gauge group. Spin indices, on the other hand, are denoted by s_1, \dots, s_m , and they need to take into account how helicities change in CDR: for gluons $s_i \equiv \mu_i \in \{1, \dots, d\}$, while for quarks $s_i \in \{1, 2\}$.

Consider the m -parton colour-space \mathcal{H}_c and helicity-space \mathcal{H}_s , and introduce an orthonormal basis in each:

$$\{|c_1, \dots, c_m\rangle\} \in \mathcal{H}_c \quad \{|s_1, \dots, s_m\rangle\} \in \mathcal{H}_s$$

Note that, as these are finite-dimensional Hilbert spaces, the non-canonical (basis-dependent) isomorphisms $\mathcal{H}_c \leftrightarrow \mathcal{H}_c^*$ and $\mathcal{H}_s \leftrightarrow \mathcal{H}_s^*$ are well-defined³.

Then, to make the colour-helicity structure of the m -parton amplitude explicit, we define it as an abstract vector in $\mathcal{H}_c \otimes \mathcal{H}_s$, so that:

$$\mathcal{A}_m^{\{c_1, \dots, c_m\}, \{s_1, \dots, s_m\}}(\{p_1, \dots, p_m\}) \equiv \langle \{c_1, \dots, c_m\}, \{s_1, \dots, s_m\} | \mathcal{A}_m(\{p_1, \dots, p_m\}) \rangle \quad (2.25)$$

with:

$$|\{c_1, \dots, c_m\}, \{s_1, \dots, s_m\}\rangle \equiv |c_1, \dots, c_m\rangle \otimes |s_1, \dots, s_m\rangle$$

Hence, it is clear that the squared amplitude summed over colours and helicities is:

$$|\mathcal{A}_m|^2 = \langle \mathcal{A}_m | \mathcal{A}_m \rangle \quad (2.26)$$

To represent colour interactions at QCD vertices, we associate to each parton i a colour charge $\mathbf{T}_i = \{T_i^a\}_{a=1, \dots, n_c^2-1}$ related to the emission of a gluon. The action of \mathbf{T}_i onto \mathcal{H}_c is defined by:

$$\langle c_1, \dots, c_i, \dots, c_m | T_i^a | b_1, \dots, b_i, \dots, b_m \rangle = \delta_{c_1, b_1} \dots T_{c_i b_i}^a \dots \delta_{c_m, b_m} \quad (2.27)$$

Thus, $\{T_{c_i b_i}^a\}_{a=1, \dots, n_c^2-1}$ form a vector with respect to the colour index a of the emitted gluon, and they are matrices in different representations of $SU(n_c)$, depending on the parton i :

- if i is a gluon, then $T_{cb}^a \equiv if^{cab}$ (adjoint representation);
- if i is a final-state quark, then $T_{\alpha\beta}^a \equiv t_{\alpha\beta}^a$ (fundamental representation), while if it is a final-state antiquark $T_{\alpha\beta}^a \equiv -t_{\alpha\beta}^a$ (conjugate of fundamental representation);
- if i is an initial-state quark, by crossing-symmetry $T_{\alpha\beta}^a \equiv -t_{\alpha\beta}^a$, while if it is an initial-state antiquark $T_{\alpha\beta}^a \equiv t_{\alpha\beta}^a$.

³Given a finite-dimensional \mathbb{K} -vector space V and a basis $\{v_i\}_{i=1, \dots, n} \subset V$, with $n = \dim_{\mathbb{K}} V$, then a basis $\{\omega^1, \dots, \omega^n\} \subset V^*$ of $V^* := \text{Hom}(V, \mathbb{K})$ is defined by $\omega^i(v_j) = \delta_j^i$, and the function $\varphi : V \rightarrow V^* : v_i \mapsto \omega^i$ is a *non-canonical isomorphism* $V \leftrightarrow V^*$.

If V is infinite-dimensional, instead, given a basis $\{v_i\}_{i \in \mathcal{I}} \subset V$, the above construction only allows to define linearly-independent subsets of V^* , which are not granted to be bases.

The algebra of these QCD colour-charge operators is easily determined. First of all, we set:

$$\mathbf{T}_i \cdot \mathbf{T}_j \equiv \sum_{a=1}^{n_c^2-1} T_i^a T_j^a \quad (2.28)$$

Then, by the action Eq. 2.27, it is clear that charges associated to different partons commute, i.e.:

$$\mathbf{T}_i \cdot \mathbf{T}_j = \mathbf{T}_j \cdot \mathbf{T}_i \quad \forall i \neq j \in \{1, \dots, m\} \quad (2.29)$$

Moreover, by Eq. 2.15, 2.28:

$$\mathbf{T}_i^2 = C_i \text{id}_{\mathcal{H}_c} \quad (2.30)$$

with $C_i \equiv C_F$ if i is a quark/antiquark and $C_i \equiv C_A$ if it is a gluon, i.e. the quadratic Casimir operators Eq. 2.16. Finally, as each vector $|\mathcal{A}_m\rangle$ is a colour-singlet, colour conservation implies:

$$\sum_{i=1}^m \mathbf{T}_i |\mathcal{A}_m\rangle = 0 \quad (2.31)$$

This allows us to partially (or fully, if $m = 2$ or $m = 3$, as in Appendix A of [28]) factorize the colour-charge algebra in terms of quadratic Casimir operators.

NSC Subtraction Scheme

The aim of the NSC subtraction scheme (SS) is to regulate the IR divergences in QCD corrections to the inclusive¹ production of jets in a hadron collider, i.e. to the process:

$$p + p \rightarrow X + N \text{ jets} \quad (3.1)$$

where X is a colour-neutral system. The hadron-scale physics can be treated separately from the parton-scale physics (see Section 1.1 of [45] and §1.1): this makes it possible for us to directly manipulate partonic cross-sections according to Eq. 1.3, where now the sum runs over all initial-state massless partons a and b which contribute to the production of the final state under consideration. Moreover, for the rest of this work we set $\mu_R = \mu_F = \mu$, where μ is the typical energy scale of the considered process.

Denoting the partons' momenta as $p_i \equiv \xi_i P_i$, $i = 1, 2$, and suppressing the explicit dependence on the running coupling and the renormalization scale, it is possible to express the LO term of Eq. 1.7 as (see §A.1 for the definition of the m -particle phase-space measure $d\Phi_m$):

$$d\hat{\sigma}_{a,b}^{(0)}(p_1, p_2) := \sum_{\mathbf{f}} \frac{\mathcal{N}}{2\hat{s}} \int d\Phi_m (2\pi)^4 \delta^{(4)}(p_{\text{in}} - p_{\text{out}}) \left| \mathcal{A}_m^{(0)}(p_1, p_2, p_X, p_{\mathbf{f}}) \right|^2 \mathcal{O}_m(p_X, p_{\mathbf{f}}) \quad (3.2)$$

where m is the number of final-state partons ($m = N$ at LO), $\hat{s} \equiv 2p_1 \cdot p_2$ is the partonic center-of-mass (CM) energy squared, $p_{\mathbf{f}}$ is the total final-state partonic momentum and the normalization factor \mathcal{N} includes all necessary symmetry factors, as well as averaging factors for initial-state colours and helicities. The sum runs over all possible partonic final states for the considered process (formalized in the next section). Moreover, \mathcal{O}_m is an IR-finite measurement function defining the observable, which ensures that the final state contains at least N resolved jets: in particular, if the energy of a final-state gluon vanishes (soft limit), or if two partons become collinear to one another (collinear limit), then $\mathcal{O}_{m+n} \rightarrow \mathcal{O}_{m+n-1}$ for $n \in \mathbb{N}$, and $\mathcal{O}_m \rightarrow 0$. Moreover, we assume that \mathcal{O}_m contains the integration measure $d\Phi_X$ for the colour-singlet. Similarly, it is possible to write the NLO corrections in Eq. 1.8 as:

$$d\hat{\sigma}_{a,b}^{\text{R}}(p_1, p_2) := \sum_{\mathbf{f}} \frac{\mathcal{N}'}{2\hat{s}} \int d\Phi_{m+1} (2\pi)^4 \delta^{(4)}(p_{\text{in}} - p_{\text{out}}) \left| \mathcal{A}_{m+1}^{(0)}(p_1, p_2, p_X, p_{\mathbf{f}}) \right|^2 \mathcal{O}_{m+1}(p_X, p_{\mathbf{f}}) \quad (3.3)$$

$$d\hat{\sigma}_{a,b}^{\text{V}}(p_1, p_2) := \sum_{\mathbf{f}} \frac{\mathcal{N}}{2\hat{s}} \frac{\alpha_s(\mu_R^2)}{2\pi} \int d\Phi_m (2\pi)^4 \delta^{(4)}(p_{\text{in}} - p_{\text{out}}) 2\Re \langle \mathcal{A}_m^{(0)} | \mathcal{A}_m^{(1)} \rangle \mathcal{O}_m(p_X, p_{\mathbf{f}}) \quad (3.4)$$

¹Inclusive jet production denotes the theoretical prediction (or experimental measurement) of the cross-section for the production of jets of given kinematics, while summing/integrating over all other final-state radiation and particles.

$$d\hat{\sigma}_{a,b}^C(p_1, p_2) := \frac{\alpha_s(\mu_R^2)}{2\pi} \frac{1}{\epsilon} \sum_c \int_0^1 \frac{dz}{z} \left[\hat{P}_{f_c, f_a}^{(0)}(z) d\hat{\sigma}_{c,b}^{(0)}(zp_1, p_2) + \hat{P}_{f_c, f_b}^{(0)}(z) d\hat{\sigma}_{a,c}^{(0)}(p_1, zp_2) \right] \quad (3.5)$$

Note that in Eq. 3.3 the final state contains $m+1$ partons. The **Altarelli-Parisi splitting kernels** $\hat{P}_{f_i, f_j}^{(0)}(z)$ are listed in §B.2, and proof of Eq. 3.5 is provided in §3.2.2.

The rest of this chapter is devoted to the extraction of IR-singularities from Eq. 3.3-3.5, proving their cancellation and providing the associated integrated counterterms.

§3.1 Nested subtraction

As suggested by the name, in the NSC SS the IR-poles of real corrections are removed sequentially, starting from those arising from soft limits and then subtracting the collinear ones from the soft-regulated terms. Note that, at NLO, this scheme is identical to the FKS scheme.

To show this procedure, we introduce some notation². First of all, we define the integrand function in Eq. 3.2-3.3 as:

$$\mathcal{F}_{a,b}[\mathcal{X}_m^n] \equiv \mathcal{N}_{\text{sym}} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_X - p_f) \left| \mathcal{A}_m^{(0)}(p_1, p_2, p_X, p_f) \right|^2 \mathcal{O}_m(p_X, p_f) \quad (3.6)$$

where \mathcal{X}_m^n is the set of m final-state partons and \mathcal{N}_{sym} is the relative symmetry factor. The index $n \in \mathcal{G}_m[a, b, X]$ enumerates all the possible QCD final states which may contribute to the partonic process: these include all combinations of flavours $\{f_i\}_{i=1,\dots,m}$ consistent with the initial state (a, b) and the colour-singlet X . In the following, we suppress the arguments of \mathcal{G}_m . The integration on the m -parton final-state phase space is defined as:

$$\langle \mathcal{F}_{a,b}[\mathcal{X}_m^n] \rangle := \mathcal{N}_{\text{avg}} \int d\Phi_m \mathcal{F}_m^{a,b}[\mathcal{X}_m^n] \quad (3.7)$$

where \mathcal{N}_{avg} is the appropriate initial-state averaging factor. Then, we can rewrite Eq. 3.2-3.3 as:

$$2\hat{s} d\hat{\sigma}_{a,b}^{(0)} = \sum_{n \in \mathcal{G}_m} \langle \mathcal{F}_{a,b}[\mathcal{X}_m^n] \rangle \quad 2\hat{s} d\hat{\sigma}_{a,b}^R = \sum_{n \in \mathcal{G}_{m+1}} \langle \mathcal{F}_{a,b}[\mathcal{X}_{m+1}^n] \rangle \quad (3.8)$$

Soft and collinear singularities are isolated through operators acting on \mathcal{F} functions: S_i denotes the limit in which the parton i becomes soft, while C_{ij} that in which the partons i and j become collinear to each other. These operators extract only the leading asymptotic behaviour of \mathcal{F} which is non-integrable in $d = 4$ dimensions; hence, if they act on quantities without non-integrable singularities, then they identically vanish (e.g. $S_i \equiv 0$ if i is a (anti)quark).

§3.1.1 Partonic sets

A delicate step is the determination of which final-state partons can become unresolved: indeed, in fixed-order perturbative QCD, the number of final-state hard partons cannot drop below the number of jets in the LO process. This means that at NLO no more than one parton can become unresolved, and this is ensured by the \mathcal{O} operators. In order to use symmetry arguments to minimize the number of unresolved partons that need to be considered, we can partition the set of final-state partons as:

$$\mathcal{X}_m^n = \mathcal{X}_m^n(g) \cup \mathcal{X}_m^n(q) \cup \mathcal{X}_m^n(\bar{q}) \cup \mathcal{X}_m^n(Q) \cup \mathcal{X}_m^n(\bar{Q})$$

²The notation is slightly different from that in [32–34]: in particular, we set $F_{\text{LM}} \mapsto \mathcal{F}$ and $\mathcal{B} \mapsto \mathcal{X}$.

which are respectively the subsets of final-state gluons, massless quarks, massless antiquarks, massive quarks and massive antiquarks. It is also useful to define the set of all massless partons:

$$\mathcal{H}_{m,0}^n \equiv \{a, b\} \cup \mathcal{X}_m^n(g) \cup \mathcal{X}_m^n(q) \cup \mathcal{X}_m^n(\bar{q}) \equiv \{a, b\} \cup \mathcal{H}_m^n$$

as we only consider massless initial-state partons. Note that $\mathcal{X}_m^n(q)$ and $\mathcal{X}_m^n(\bar{q})$ can be further partitioned into sets of definite massless quark flavours, and the same can be done with $\mathcal{X}_m^n(Q)$ and $\mathcal{X}_m^n(\bar{Q})$ with massive quark flavours.

For the remainder of this chapter, we set $\mathcal{X}_m^n(Q) = \mathcal{X}_m^n(\bar{Q}) = \emptyset$: this means that $\mathcal{H}_m^n \equiv \mathcal{X}_m^n$, however we keep this redundant notation in all the equations, as this will set the stage for the next chapter.

For NLO real emissions we consider an additional parton, i.e. a final state \mathcal{X}_{m+1}^n with $n \in \mathcal{G}_{m+1}$. To precisely identify potentially-unresolved partons, we first consider a partition of unity such that:

$$\sum_{i \in \mathcal{H}_{m+1}^n} \Delta^i = 1 \quad : \quad S_i \Delta^j = \delta_i^j \quad \wedge \quad C_{ij} \Delta^k = \begin{cases} 0 & i, j \neq k \\ 1 & i = k, j \in \{a, b\} \\ z_{k,j} & i = k, j \in \mathcal{H}_{m+1}^n - \{i\} \end{cases} \quad (3.9)$$

with $z_{k,j} \equiv E_k/(E_k + E_j)$. An explicit construction of these damping factors is given in §A.2. It is clear that a term multiplied by Δ^i vanishes if any parton other than i becomes unresolved, thus this partition allows for the extraction of single potentially-unresolved partons. We can thus write the real-emission contribution as:

$$2\hat{s} d\hat{\sigma}_{a,b}^R = \sum_{n \in \mathcal{G}_{m+1}} \sum_{i \in \mathcal{H}_{m+1}^n} \langle \Delta^i \mathcal{F}_{ab}[\mathcal{X}_{m+1}^n] \rangle$$

We can relabel the potentially-unresolved parton i in each term as \mathbf{m}_{f_i} : then, for each allowed massless³ flavour f , there are N_f equal terms, where N_f is the number of final-state partons of flavour f . We can account for the cancellation of these factors with symmetry factors defining $\mathcal{X}_m^n(\mathbf{m}_f) \equiv \mathcal{X}_{m+1}^n - \{\mathbf{m}_f\}$. Hence, we impose that the symmetry factors of $\mathcal{F}_{a,b}[\mathcal{X}_m^n(\mathbf{m}_f)]$ are determined ignoring \mathbf{m}_f (i.e. implicitly multiplying by N_f), but with the convention that the amplitude in $\mathcal{F}_{a,b}[\mathcal{X}_m^n(\mathbf{m}_f)]$ still contains the potentially-unresolved parton \mathbf{m}_f . Therefore:

$$\begin{aligned} 2\hat{s} d\hat{\sigma}_{a,b}^R = & \sum_{n \in \mathcal{G}_{m+1}(g)} \langle \Delta^{\mathbf{m}} \mathcal{F}_{a,b}[\mathcal{X}_m^n(\mathbf{m}_g)] \rangle + \sum_{\rho=1}^{n_f} \sum_{n \in \mathcal{G}_{m+1}(q_\rho)} \langle \Delta^{\mathbf{m}} \mathcal{F}_{a,b}[\mathcal{X}_m^n(\mathbf{m}_{q_\rho})] \rangle \\ & + \sum_{\rho=1}^{n_f} \sum_{n \in \mathcal{G}_{m+1}(\bar{q}_\rho)} \langle \Delta^{\mathbf{m}} \mathcal{F}_{a,b}[\mathcal{X}_m^n(\mathbf{m}_{\bar{q}_\rho})] \rangle \end{aligned} \quad (3.10)$$

where $\mathcal{G}_m(f) \subset \mathcal{G}_m : N_f \geq 1$ denotes the subset of possible final states with at least one parton of flavour f . The subscript f in \mathbf{m}_f is suppressed when implicitly understood.

We are now ready to apply the nested subtraction procedure introduced in [31]. In particular, for each term we rewrite the identity operator as:

$$\text{id} = S_{\mathbf{m}} + \sum_{i \in \mathcal{H}_{m,0}^n(\mathbf{m})} \bar{S}_{\mathbf{m}} C_{i\mathbf{m}} + O_{\text{NLO}}^{\mathbf{m}} \quad O_{\text{NLO}}^{\mathbf{m}} := \sum_{i \in \mathcal{H}_{m,0}^n(\mathbf{m})} \bar{S}_{\mathbf{m}} \bar{C}_{i\mathbf{m}} \omega^{\mathbf{m}i} \quad (3.11)$$

³Massive partons cannot become unresolved, as they do not determine soft or collinear singularities. Moreover, with a slight notation abuse, we consider the gluon as a “partonic flavour”.

where we defined the notation for generic operators $\overline{\mathcal{O}} \equiv \text{id} - \mathcal{O}$ and introduced an angular partition of unity (see §A.2):

$$\sum_{i \in \mathcal{H}_{m,0}^n(\mathbf{m})} \omega^{\mathbf{m}i} = 1 \quad : \quad \mathcal{C}_{j\mathbf{m}} \omega^{\mathbf{m}i} = \delta_j^i \quad (3.12)$$

It now remains to understand how the operators in Eq. 3.11 act on the \mathcal{F} functions and how the partonic set $\mathcal{X}_m^n(\mathbf{m})$ changes when \mathbf{m} effectively becomes unresolved.

§3.1.2 Soft limits

We first consider the soft limit of each term in Eq. 3.10. For the last two terms this is trivial, since quarks do not determine soft singularities:

$$\mathcal{S}_{\mathbf{m}_q} = \mathcal{S}_{\mathbf{m}_{\bar{q}}} \equiv 0$$

Thus, the only soft singularities come from the first term, when a gluon becomes unresolved. In this limit, the factorization of the amplitude is found to be (see [28]):

$$\mathcal{S}_{\mathbf{m}_g} \mathcal{F}_{a,b}[\mathcal{X}_m^n(\mathbf{m}_g)] = -4\pi\alpha_{s,b} \sum_{i,j \in \mathcal{X}_{m,0}^n} \mathcal{S}_{i,j}(\mathbf{T}_i \cdot \mathbf{T}_j) \mathcal{F}_{a,b}[\mathcal{X}_m^n] \quad (3.13)$$

where $\mathcal{X}_{m,0}^n = \{a, b\} \cup \mathcal{X}_m^n$ and the **eikonal factor** reads:

$$\mathcal{S}_{i,j} \equiv \frac{p_i \cdot p_j}{(p_i \cdot p_{\mathbf{m}})(p_j \cdot p_{\mathbf{m}})} \quad (3.14)$$

To perform the integration on the unresolved phase space, we extract the $[dp_{\mathbf{m}}]$ measure from $d\Phi_{m+1}$:

$$\langle \mathcal{S}_{\mathbf{m}} \Delta^{\mathbf{m}} \mathcal{F}_{a,b}[\mathcal{X}_m^n(\mathbf{m}_g)] \rangle = -4\pi\alpha_{s,b} \sum_{i,j \in \mathcal{X}_{m,0}^n} (\mathbf{T}_i \cdot \mathbf{T}_j) \left\langle \int [dp_{\mathbf{m}}] \mathcal{S}_{i,j} \mathcal{F}_{a,b}[\mathcal{X}_m^n] \right\rangle$$

Note that $\mathcal{S}_{i,i} = 0$, hence the sum runs on $i \neq j \in \mathcal{X}_{m,0}^n$. Moreover, the only factor dependent on $p_{\mathbf{m}}$ is the eikonal factor, as $\mathcal{S}_{\mathbf{m}}$ removes $p_{\mathbf{m}}$ from the momentum-conserving delta function in $\mathcal{F}_{a,b}[\mathcal{X}_m^n(\mathbf{m}_g)]$: for this reason, we need to introduce an upper cut-off \mathcal{E} in order to constrain the momentum $p_{\mathbf{m}}$. Hence, we can perform the integration explicitly:

$$\begin{aligned} \int [dp_{\mathbf{m}}] \mathcal{S}_{i,j} &= \int [dp_{\mathbf{m}}] \frac{p_i \cdot p_j}{(p_i \cdot p_j)(p_j \cdot p_{\mathbf{m}})} = \int_0^{\mathcal{E}} \frac{dE_{\mathbf{m}}}{E_{\mathbf{m}}^{1+2\epsilon}} \int_{\mathbb{S}^{2-2\epsilon}} \frac{d\Omega_{2-2\epsilon}}{2(2\pi)^{3-2\epsilon}} \frac{\rho_{ij}}{\rho_{i\mathbf{m}}\rho_{j\mathbf{m}}} \\ &= -\frac{\mathcal{E}^{-2\epsilon}}{2\epsilon} \rho_{ij} \frac{1}{8\pi^2} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} 2^{-1-2\epsilon} \frac{\Gamma^2(-\epsilon)}{\Gamma(-2\epsilon)} {}_2F_1(1, 1, 1-\epsilon, 1-\eta_{ij}) \\ &= \frac{1}{\epsilon^2} \frac{[\alpha_s]}{4\pi\alpha_{s,b}} \left(\frac{2\mathcal{E}}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \eta_{ij} {}_2F_1(1, 1, 1-\epsilon, 1-\eta_{ij}) \end{aligned}$$

where ${}_2F_1(a, b, c, z)$ is the Gauss hypergeometric function and we made use of the short-hands defined in Eq. B.2 and the angular integral from Appendix G.3 of [19]. It is thus possible to write the integrated soft-counterterm as:

$$\sum_{n \in \mathcal{G}_{m+1}(g)} \langle \mathcal{S}_{\mathbf{m}} \Delta^{\mathbf{m}} \mathcal{F}_{a,b}[\mathcal{X}_m^n(\mathbf{m}_g)] \rangle = [\alpha_s] \sum_{n \in \mathcal{G}_m} \langle I_S(\epsilon) \mathcal{F}_{a,b}[\mathcal{X}_m^n] \rangle \quad (3.15)$$

where the **integrated soft operator** is defined as:

$$I_S(\epsilon) := -\frac{1}{\epsilon^2} \left(\frac{2\mathcal{E}}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \sum_{i \neq j \in \mathcal{X}_{m,0}^n} \eta_{ij} (\mathbf{T}_i \cdot \mathbf{T}_j) {}_2F_1(1, 1, 1-\epsilon, 1-\eta_{ij}) \quad (3.16)$$

It is important to note that $I_S(\epsilon) \sim \epsilon^{-2}$.

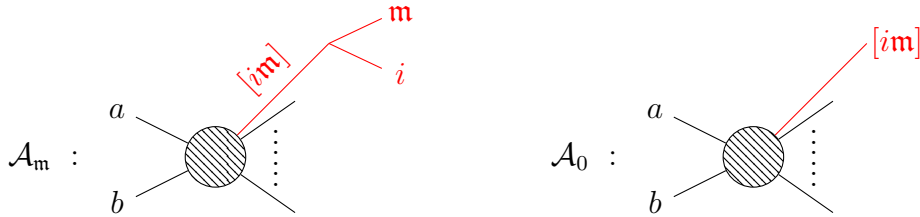
§3.1.3 Collinear limits

Collinear limits are more delicate to analyze, due to the fact that each potentially-unresolved parton can become collinear to any other parton in $\mathcal{H}_{m,0}^n(\mathbf{m})$. To consider a single collinear limit at a time, i.e. a single parton pair becoming collinear, we use the partition Eq. 3.12. In particular, we are interested in the extraction of hard-collinear singularities stemming from terms of the form $\bar{S}_m C_{im}$, which in the case of unresolved quarks coincides with C_{im} .

§3.1.3.1 Generalized anomalous dimensions

The factorization of the amplitude in a collinear limit can be found in [28], and it depends on whether the unresolved final-state parton becomes collinear to an initial- or final-state parton.

Final-state collinear limit Consider the case of a final-state parton \mathbf{m} of flavour f_m becoming collinear to another final-state parton i of flavour f_i . Then, we set:



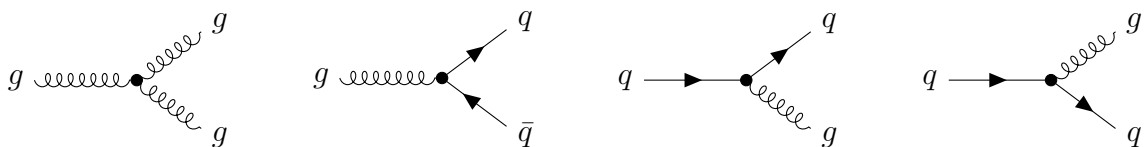
Note that we only consider amplitudes corresponding to a singular limit, and these must proceed through an intermediate state $[im]$: for this reason, we ignore the case of i and \mathbf{m} being produced directly from the hard process. The factorization of the amplitude reads:

$$C_{im} \mathcal{A}_m = -\frac{8\pi\alpha_{s,b}}{(p_i - p_m)^2} P_{f_{[im]}f_i}(z) \mathcal{A}_0 \quad (3.17)$$

where $P_{f_{[im]}f_i}(z)$ is the **Altarelli-Parisi splitting function** associated to the splitting process $[im] \rightarrow i + \mathbf{m}$ and z is the momentum fraction carried by the parton i , i.e.:

$$z \equiv 1 - \frac{E_m}{E_{[im]}} \quad (3.18)$$

The possible splittings are determined by the QCD interaction vertices (see §2.1.3) and are listed in Eq. B.8-B.11:



and respective charge-conjugates (splitting functions do not distinguish between quarks and antiquarks). Note that we represented $[im]$ on the left, m on top and i on bottom.

Using Eq. 3.17 we can derive a general expression for the integrated final-state collinear counterterm:

$$\begin{aligned} \langle C_{im} \Delta^m \mathcal{F}_{a,b}[\mathcal{X}_m^n(m)] \rangle &= \left\langle \int [dp_i][dp_m] \frac{4\pi\alpha_{s,b}}{p_i \cdot p_m} z P_{f_{[im]}f_i}(z) \mathcal{F}_{a,b}[\mathcal{X}_m^{n'}](p_{[im]}) \right\rangle \\ &= \left\langle \int_{(\mathbb{S}^{2-2\epsilon})^2} \frac{d\Omega_{2-2\epsilon}^2}{(2(2\pi)^{3-2\epsilon})^2} \int_0^\mathcal{E} \frac{dE_i}{E_i^{-1+2\epsilon}} \int_0^\mathcal{E} \frac{dE_m}{E_m^{-1+2\epsilon}} \frac{4\pi\alpha_{s,b}}{E_{[im]}^2 \rho_{im}} \frac{P_{f_{[im]}f_i}(z)}{1-z} \mathcal{F}_{a,b}[\mathcal{X}_m^{n'}](p_{[im]}) \right\rangle \\ &= -\frac{[\alpha_s]}{\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \left\langle \int [dp_{[im]}] \left(\frac{2E_{[im]}}{\mu} \right)^{-2\epsilon} \int_{z_m}^1 dz \frac{z P_{f_{[im]}f_i}(z)}{[z(1-z)]^{2\epsilon}} \mathcal{F}_{a,b}[\mathcal{X}_m^{n'}](p_{[im]}) \right\rangle \end{aligned}$$

where we performed a change of variables $(E_i, E_m) \mapsto (E_{[im]}, z) : E_i = zE_{[im]}, E_m = (1-z)E_{[im]}$. Note that we made the dependence of $\mathcal{F}_{a,b}[\mathcal{X}_m^{n'}]$ on $p_{[im]} = z^{-1}p_i$ explicit, making it clear that the \mathcal{F} function vanishes when $z < 0$, which is the case for $z \in [z_m, 0)$ as $z_m \equiv 1 - \mathcal{E}/E_{[im]} < 0$. Including the soft-regulated operator \bar{S}_m , the hard-collinear counterterm reads:

$$\begin{aligned} \langle \bar{S}_m C_{im} \Delta^m \mathcal{F}_{a,b}[\mathcal{X}_m^n(m)] \rangle &= -\frac{[\alpha_s]}{\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \left\langle \int [dp_{[im]}] \left(\frac{2E_{[im]}}{\mu} \right)^{-2\epsilon} \int_{z_m}^1 dz \bar{S}_z \frac{z P_{f_{[im]}f_i}(z)}{[z(1-z)]^{2\epsilon}} \mathcal{F}_{a,b}[\mathcal{X}_m^{n'}](p_{[im]}) \right\rangle \end{aligned}$$

where $S_z \equiv \lim_{z \rightarrow 1}$ is the S_m operator expressed in terms of the z variable. Since S_z only extracts the singular part, and by Eq. B.8-B.11, it is clear that:

$$S_z P_{f_{[im]}f_i}(z) = \frac{2}{1-z} \mathbf{T}_{[im]}^2 \delta_{f_{[im]},f_i} \quad (3.19)$$

We then write $[z_m, 1] = [z_m, 0) \cup [0, 1]$, and note that the region $[z_m, 0)$ gives a non-zero contribution in the soft limit. It only remains to evaluate the following integral:

$$\begin{aligned} \int_{z_m}^0 dz \bar{S}_z \frac{z P_{f_{[im]}f_i}(z)}{[z(1-z)]^{2\epsilon}} \mathcal{F}_{a,b}[\mathcal{X}_m^{n'}](z^{-1}p_i) &= -2\mathbf{T}_{[im]}^2 \delta_{f_{[im]},f_i} \int_0^{z_m} dz (1-z)^{-1-2\epsilon} \mathcal{F}_{a,b}[\mathcal{X}_m^{n'}](p_i) \\ &= \mathbf{T}_{[im]}^2 \delta_{f_{[im]},f_i} \frac{1 - e^{-2\epsilon L_i}}{\epsilon} \mathcal{F}_{a,b}[\mathcal{X}_m^{n'}] \end{aligned}$$

where we set $L_i \equiv \log(\mathcal{E}/E_i)$ and suppressed the dependence of the \mathcal{F} function on $p_i = S_z p_{[im]}$. Combining the above results, the integrated final-state hard-collinear counterterm can be expressed as:

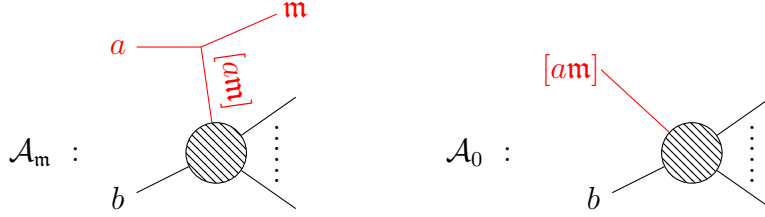
$$\langle \bar{S}_m C_{im} \Delta^m \mathcal{F}_{a,b}[\mathcal{X}_m^n(m)] \rangle = [\alpha_s] \left\langle \frac{\Gamma_{i,f_{[im]} \rightarrow f_i f_m}}{\epsilon} \mathcal{F}_{a,b}[\mathcal{X}_m^{n'}] \right\rangle \quad (3.20)$$

where $n' \in \mathcal{G}_m$ and the generalized process-dependent final-state anomalous dimension is defined as:

$$\Gamma_{i,f_{[im]} \rightarrow f_i f_m} := -\left(\frac{2E_i}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \left[\int_0^1 dz \bar{S}_z \frac{z P_{f_{[im]}f_i}(z)}{[z(1-z)]^{2\epsilon}} - \delta_{f_{[im]},f_i} \mathbf{T}_{[im]}^2 \frac{1 - e^{-2\epsilon L_i}}{\epsilon} \right] \quad (3.21)$$

The process-dependence of this generalized anomalous dimension, as opposed to the initial-state one discussed below, is due to the z factor coming from the damping factor Δ^m (Eq. 3.9).

Initial-state collinear limit Now, consider a final-state parton \mathbf{m} of flavour $f_{\mathbf{m}}$ becoming collinear to an initial-state parton a of flavour f_a . In this case, we set:



The factorization of the amplitude reads:

$$\mathcal{C}_{am}\mathcal{A}_m = -\frac{8\pi\alpha_{s,b}}{(p_a - p_m)^2} \frac{1}{z} P_{f_{[am]}f_a,i}(z) \mathcal{A}_0 \quad (3.22)$$

where the Altarelli-Parisi initial-state splitting functions are defined in Eq. B.12-B.15 and depend on the energy fraction:

$$z \equiv 1 - \frac{E_m}{E_a} \quad (3.23)$$

associated to the splitting process $a \rightarrow [am] + \mathbf{m}$. The integrated initial-state collinear counterterm is found analogously to the final-state one:

$$\langle \mathcal{C}_{am} \Delta^m \mathcal{F}_{a,b}[\mathcal{X}_m^n(\mathbf{m})] \rangle = -\frac{[\alpha_s]}{\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \left\langle \left(\frac{2E_{[am]}}{\mu} \right)^{-2\epsilon} \int_{z_m}^1 dz \frac{P_{f_{[am]}f_a,i}(z)}{z(1-z)^{2\epsilon}} \mathcal{F}_{[am],b}[\mathcal{X}_m^{n'}(p_{[am]})] \right\rangle$$

where $p_{[am]} = zp_a$. As in Appendix A.3 of [34], we define new splitting functions:

$$\mathcal{P}_{f_{[am]}f_a}(z, E_a) := \bar{S}_z \frac{P_{f_{[am]}f_a,i}(z)}{(1-z)^{2\epsilon}} - \delta_{f_{[am]}f_a} \mathbf{T}_{[am]}^2 \frac{1 - e^{-2\epsilon L_a}}{\epsilon} \delta(1-z) \quad (3.24)$$

which can be conveniently rewritten as:

$$-\mathcal{P}_{f_{[am]}f_a}(z, E_a) = \delta_{f_{[am]}f_a} \delta(1-z) \left[\gamma_a + \mathbf{T}_a^2 \frac{1 - e^{-2\epsilon L_a}}{\epsilon} \right] + \left[-\hat{P}_{f_{[am]}f_a}^{(0)}(z) + \epsilon \mathcal{P}_{f_{[am]}f_a}^{\text{fin}}(z) \right] \quad (3.25)$$

where γ_a is the anomalous dimension of the parton a and $\mathcal{P}_{f_{[am]}f_a}^{\text{fin}}$ is the ϵ -expansion of the function $-\mathcal{P}_{f_{[am]}f_a}(z, 0)$ starting at $o(\epsilon)$. This relation allows us to get an explicit expression for the integrated initial-state hard-collinear counterterm:

$$\langle \bar{S}_m \mathcal{C}_{am} \Delta^m \mathcal{F}_{a,b}[\mathcal{X}_m^n(\mathbf{m})] \rangle = \frac{[\alpha_s]}{\epsilon} \left\langle \delta_{f_{[am]}f_a} \Gamma_{a,f_a} \mathcal{F}_{[am],b}[\mathcal{X}_m^{n'}] \right\rangle + \frac{[\alpha_s]}{\epsilon} \left\langle \mathcal{P}_{f_{[am]}f_a}^{\text{gen}} \otimes \mathcal{F}_{[am],b}[\mathcal{X}_m^{n'}] \right\rangle \quad (3.26)$$

where $n' \in \mathcal{G}_m[[am], b, X]$ and the **generalized initial-state anomalous dimension** and the generalized splitting functions are defined as:

$$\Gamma_{a,f_a} := \left(\frac{2E_a}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \left[\gamma_a + \mathbf{T}_a^2 \frac{1 - e^{-2\epsilon L_a}}{\epsilon} \right] \quad (3.27)$$

$$\mathcal{P}_{f_{[am]}f_a}^{\text{gen}}(z) := \left(\frac{2E_a}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \left[-\hat{P}_{f_{[am]}f_a}^{(0)}(z) + \epsilon \mathcal{P}_{f_{[am]}f_a}^{\text{fin}}(z) \right] \quad (3.28)$$

and the Mellin convolution is defined as:

$$\mathcal{P}_{f_{[am]}f_a}^{\text{gen}} \otimes \mathcal{F}_{[am],b}[\mathcal{X}_m^{n'}](p_a) \equiv \int_0^1 \frac{dz}{z} \mathcal{P}_{f_{[am]}f_a}^{\text{gen}}(z) \mathcal{F}_{[am],b}[\mathcal{X}_m^{n'}](zp_a) \quad (3.29)$$

§3.1.3.2 Hard-collinear limits

The only difficulty left is to combine the generalized anomalous dimensions arising from the various terms in Eq. 3.10.

We begin by considering an unresolved gluon, i.e. applying the hard-collinear operator $\bar{S}_{m_g} C_{im_g}$. Since the potentially-unresolved gluon can be clustered with, or emitted from, any parton without changing the latter's flavour, under the action of this hard-collinear operator $\mathcal{X}_m^n(\mathbf{m}_g) \mapsto \mathcal{X}_m^n$, i.e. the resolved partons remain unchanged. The symmetry factors in the \mathcal{F} function thus do not change, so, from Eq. 3.20, 3.26:

$$\sum_{n \in \mathcal{G}_{m+1}(g)} \langle \bar{S}_{m_g} C_{am} \Delta^m \mathcal{F}_{a,b}[\mathcal{X}_m^n(\mathbf{m}_g)] \rangle = \sum_{n \in \mathcal{G}_m} \frac{[\alpha_s]}{\epsilon} [\langle \Gamma_{a,fa} \mathcal{F}_{a,b}[\mathcal{X}_m^n] \rangle + \langle \mathcal{P}_{fafa}^{\text{gen}} \otimes \mathcal{F}_{a,b}[\mathcal{X}_m^n] \rangle] \quad (3.30)$$

$$\sum_{n \in \mathcal{G}_{m+1}(g)} \sum_{i \in \mathcal{H}_m^n(\mathbf{m}_g)} \langle \bar{S}_{m_g} C_{im} \Delta^m \mathcal{F}_{a,b}[\mathcal{X}_m^n] \rangle = \sum_{n \in \mathcal{G}_m} \sum_{i \in \mathcal{H}_m^n} \frac{[\alpha_s]}{\epsilon} \langle \Gamma_{i,fi \rightarrow f_{ig}} \mathcal{F}_{a,b}[\mathcal{X}_m^n] \rangle \quad (3.31)$$

We now move to the case of an unresolved quark, i.e. we consider the hard-collinear operator $\bar{S}_{m_{q\rho}} C_{im_{q\rho}} \equiv C_{im_{q\rho}}$. In this case, IR singularities arise when $\mathbf{m}_{q\rho}$ becomes collinear to:

- an initial-state gluon ($a_g \rightarrow \mathbf{m}_{q\rho} + [am]_{\bar{q}\rho}$), in which case $\mathcal{G}_{m+1}(q\rho) \mapsto \mathcal{G}_m[\bar{q}\rho, b, X]$ and $\mathcal{X}_m^n(\mathbf{m}_{q\rho}) \mapsto \mathcal{X}_m^{n'}$ composed of the same resolved partons, so from Eq. 3.26:

$$\sum_{n \in \mathcal{G}_{m+1}(q\rho)} \delta_{fa,g} \langle \bar{S}_{m_g} C_{am} \Delta^m \mathcal{F}_{a,b}[\mathcal{X}_m^n(\mathbf{m}_{q\rho})] \rangle = \sum_{n \in \mathcal{G}_m} \frac{[\alpha_s]}{\epsilon} \langle \mathcal{P}_{\bar{q}g}^{\text{gen}} \otimes \mathcal{F}_{\bar{q},b}[\mathcal{X}_m^n] \rangle$$

where we adopt the convention that the arguments of \mathcal{G}_m are determined by the \mathcal{F} function in the sum. Note that the symmetry factors of \mathcal{F} functions are solely determined by final-state resolved partons, hence initial-state hard-collinear limits always leave symmetry factors unchanged (independently of f_a and f_m); however, initial-state spin degrees of freedom and colour factors do change, and this is absorbed in the \mathcal{P}^{gen} function.

- an initial-state quark of the same flavour ($a_{q\rho} \rightarrow \mathbf{m}_{q\rho} + [am]_g$), in which case $\mathcal{G}_{m+1}(q\rho) \mapsto \mathcal{G}_m[g, b, X]$ and $\mathcal{X}_m^n(\mathbf{m}_{q\rho}) \mapsto \mathcal{X}_m^{n'}$ composed of the same resolved partons, so from Eq. 3.26:

$$\sum_{n \in \mathcal{G}_{m+1}(q\rho)} \delta_{fa,q\rho} \langle \bar{S}_{m_g} C_{am} \Delta^m \mathcal{F}_{a,b}[\mathcal{X}_m^n(\mathbf{m}_{q\rho})] \rangle = \sum_{n \in \mathcal{G}_m} \frac{[\alpha_s]}{\epsilon} \langle \mathcal{P}_{gq}^{\text{gen}} \otimes \mathcal{F}_{g,b}[\mathcal{X}_m^n] \rangle$$

- a final-state gluon ($[im]_{q\rho} \rightarrow \mathbf{m}_{q\rho} + i_g$), in which case $\mathcal{G}_{m+1}(q\rho) \mapsto \mathcal{G}_m$ and $\mathcal{X}_m^n(\mathbf{m}_{q\rho}) \mapsto \mathcal{X}_m^{n'}$ with one less gluon and one more quark of flavour ρ , hence from Eq. 3.20:

$$\sum_{n \in \mathcal{G}_{m+1}(q\rho)} \sum_{i \in \mathcal{H}_m^n(\mathbf{m}_{q\rho})} \delta_{fi,g} \langle \bar{S}_{m_g} C_{im} \Delta^m \mathcal{F}_{a,b}[\mathcal{X}_m^n(\mathbf{m}_{q\rho})] \rangle = \sum_{n \in \mathcal{G}_m} \sum_{i \in \mathcal{H}_m^n} \delta_{fi,q\rho} \frac{[\alpha_s]}{\epsilon} \langle \Gamma_{i,q \rightarrow gq} \mathcal{F}_{a,b}[\mathcal{X}_m^n] \rangle$$

A clarification on symmetry factors is now needed. Assuming that $\mathcal{X}_m^n(\mathbf{m}_{q\rho})$ contains N_g gluons and $N_{q\rho}$ quarks of flavour ρ (always not including $\mathbf{m}_{q\rho}$), then the symmetry factor is $\mathcal{N}_{\text{sym}} \sim 1/(N_g! N_{q\rho}!)$. On the other hand, $\mathcal{X}_m^{n'}$ contains $N_g - 1$ gluons and $N_{q\rho} + 1$ quarks of flavour ρ , hence there is a mismatch of $\mathcal{N}'_{\text{sym}} = \mathcal{N}_{\text{sym}} \cdot N_g / (N_{q\rho} + 1)$. This is easily understood, as the i -sum on the left-hand side gives an N_g factor, while that on the right-hand side gives an $N_{q\rho} + 1$ factor.

- a final-state antiquark of the same flavour ($[i\mathbf{m}]_g \rightarrow \mathbf{m}_{q\rho} + i_{\bar{q}\rho}$), in which case $\mathcal{G}_{m+1}(q\rho) \mapsto \mathcal{G}_m$ and $\mathcal{X}_m^n(\mathbf{m}_{q\rho}) \mapsto \mathcal{X}_m^{n'}$ with one less antiquark of flavour ρ and one more gluon, hence from Eq. 3.20:

$$\sum_{n \in \mathcal{G}_{m+1}(q\rho)} \sum_{i \in \mathcal{H}_m^n(\mathbf{m}_{q\rho})} \delta_{f_i, \bar{q}\rho} \langle \bar{\mathcal{S}}_m \mathcal{C}_{im} \Delta^m \mathcal{F}_{a,b}[\mathcal{X}_m^n(\mathbf{m}_{q\rho})] \rangle = \sum_{n \in \mathcal{G}_m} \sum_{i \in \mathcal{H}_m^n} \delta_{f_i, g} \langle \Gamma_{i, g \rightarrow \bar{q}q} \mathcal{F}_{a,b}[\mathcal{X}_m^n] \rangle$$

Putting everything together, we find:

$$\begin{aligned} \sum_{n \in \mathcal{G}_{m+1}(q\rho)} \sum_{i \in \mathcal{H}_{m,0}^n(\mathbf{m}_{q\rho})} \langle \bar{\mathcal{S}}_m \mathcal{C}_{im} \Delta^m \mathcal{F}_{a,b}[\mathcal{X}_m^n(\mathbf{m}_{q\rho})] \rangle &= (a \leftrightarrow b) + \sum_{n \in \mathcal{G}_m} \frac{[\alpha_s]}{\epsilon} \langle \delta_{f_a, g} \mathcal{P}_{\bar{q}g}^{\text{gen}} \otimes \mathcal{F}_{\bar{q}\rho, b}[\mathcal{X}_m^n] \rangle + \\ &+ \sum_{n \in \mathcal{G}_m} \frac{[\alpha_s]}{\epsilon} \langle \delta_{f_a, q\rho} \mathcal{P}_{gq}^{\text{gen}} \otimes \mathcal{F}_{g, b}[\mathcal{X}_m^n] \rangle + \sum_{n \in \mathcal{G}_m} \sum_{i \in \mathcal{H}_m^n} \frac{[\alpha_s]}{\epsilon} \langle (\delta_{f_i, q\rho} \Gamma_{i, q \rightarrow gq} + \delta_{f_i, g} \Gamma_{i, g \rightarrow \bar{q}q}) \mathcal{F}_{a, b}[\mathcal{X}_m^n] \rangle \end{aligned} \quad (3.32)$$

Finally, we note that our analysis remains unchanged if we consider an antiquark becoming unresolved, with the result being expressed using charge-conjugated splittings:

$$\begin{aligned} \sum_{n \in \mathcal{G}_{m+1}(\bar{q}\rho)} \sum_{i \in \mathcal{H}_{m,0}^n(\mathbf{m}_{\bar{q}\rho})} \langle \bar{\mathcal{S}}_m \mathcal{C}_{im} \Delta^m \mathcal{F}_{a,b}[\mathcal{X}_m^n(\mathbf{m}_{\bar{q}\rho})] \rangle &= (a \leftrightarrow b) + \sum_{n \in \mathcal{G}_m} \frac{[\alpha_s]}{\epsilon} \langle \delta_{f_a, g} \mathcal{P}_{qg}^{\text{gen}} \otimes \mathcal{F}_{q\rho, b}[\mathcal{X}_m^n] \rangle + \\ &+ \sum_{n \in \mathcal{G}_m} \frac{[\alpha_s]}{\epsilon} \langle \delta_{f_a, \bar{q}\rho} \mathcal{P}_{g\bar{q}}^{\text{gen}} \otimes \mathcal{F}_{g, b}[\mathcal{X}_m^n] \rangle + \sum_{n \in \mathcal{G}_m} \sum_{i \in \mathcal{H}_m^n} \frac{[\alpha_s]}{\epsilon} \langle (\delta_{f_i, \bar{q}\rho} \Gamma_{i, \bar{q} \rightarrow g\bar{q}} + \delta_{f_i, g} \Gamma_{i, g \rightarrow q\bar{q}}) \mathcal{F}_{a, b}[\mathcal{X}_m^n] \rangle \end{aligned} \quad (3.33)$$

§3.1.3.3 Collinear operator

All the integrated hard-collinear counterterms discussed above can be combined into a convenient expression. First of all, putting everything together:

$$\begin{aligned} \sum_{n \in \mathcal{G}_{m+1}(g)} \sum_{i \in \mathcal{H}_{m,0}^n(\mathbf{m}_g)} \langle \bar{\mathcal{S}}_m \mathcal{C}_{im} \Delta^m \mathcal{F}_{a,b}[\mathcal{X}_m^n(\mathbf{m}_g)] \rangle &+ \sum_{\rho=1}^{n_f} \sum_{n \in \mathcal{G}_{m+1}(q\rho)} \sum_{i \in \mathcal{H}_{m,0}^n(\mathbf{m}_{q\rho})} \langle \bar{\mathcal{S}}_m \mathcal{C}_{im} \Delta^m \mathcal{F}_{a,b}[\mathcal{X}_m^n(\mathbf{m}_{q\rho})] \rangle \\ &+ \sum_{\rho=1}^{n_f} \sum_{n \in \mathcal{G}_{m+1}(\bar{q}\rho)} \sum_{i \in \mathcal{H}_{m,0}^n(\mathbf{m}_{\bar{q}\rho})} \langle \bar{\mathcal{S}}_m \mathcal{C}_{im} \Delta^m \mathcal{F}_{a,b}[\mathcal{X}_m^n(\mathbf{m}_{\bar{q}\rho})] \rangle \\ &= \frac{[\alpha_s]}{\epsilon} \sum_{n \in \mathcal{G}_m} \left[[\langle \Gamma_{a, f_a} \mathcal{F}_{a,b}[\mathcal{X}_m^n] \rangle + \langle \mathcal{P}_{f_a f_a}^{\text{gen}} \otimes \mathcal{F}_{a,b}[\mathcal{X}_m^n] \rangle + (a \leftrightarrow b)] + \right. \\ &+ \sum_{\rho=1}^{n_f} [\langle \delta_{f_a, g} \mathcal{P}_{\bar{q}g}^{\text{gen}} \otimes \mathcal{F}_{\bar{q}\rho, b}[\mathcal{X}_m^n] \rangle + \langle \delta_{f_a, q\rho} \mathcal{P}_{gq}^{\text{gen}} \otimes \mathcal{F}_{g, b}[\mathcal{X}_m^n] \rangle + \langle \delta_{f_a, g} \mathcal{P}_{qg}^{\text{gen}} \otimes \mathcal{F}_{q\rho, b}[\mathcal{X}_m^n] \rangle \\ &\quad \left. + \langle \delta_{f_a, \bar{q}\rho} \mathcal{P}_{g\bar{q}}^{\text{gen}} \otimes \mathcal{F}_{g, b}[\mathcal{X}_m^n] \rangle + (a \leftrightarrow b)] \right. \\ &+ \sum_{i \in \mathcal{X}_m^n(g)} \langle \Gamma_{i, g \rightarrow gg} \mathcal{F}_{a,b}[\mathcal{X}_m^n] \rangle + \sum_{i \in \mathcal{X}_m^n(q)} \langle \Gamma_{i, q \rightarrow qg} \mathcal{F}_{a,b}[\mathcal{X}_m^n] \rangle + \sum_{i \in \mathcal{X}_m^n(\bar{q})} \langle \Gamma_{i, \bar{q} \rightarrow q\bar{q}} \mathcal{F}_{a,b}[\mathcal{X}_m^n] \rangle + \\ &\left. + 2n_f \sum_{i \in \mathcal{X}_m^n(g)} \langle \Gamma_{i, g \rightarrow \bar{q}q} \mathcal{F}_{a,b}[\mathcal{X}_m^n] \rangle + \sum_{i \in \mathcal{X}_m^n(q)} \langle \Gamma_{i, q \rightarrow gq} \mathcal{F}_{a,b}[\mathcal{X}_m^n] \rangle + \sum_{i \in \mathcal{X}_m^n(\bar{q})} \langle \Gamma_{i, \bar{q} \rightarrow g\bar{q}} \mathcal{F}_{a,b}[\mathcal{X}_m^n] \rangle \right] \end{aligned}$$

where we used the fact that generalized splitting functions and anomalous dimensions do not depend on quark/antiquark flavours and we have included terms arising from collinear limits with respect to the initial-state parton b . Collectively, Mellin convolutions can be grouped as:

$$\frac{[\alpha_s]}{\epsilon} \sum_c \sum_{n \in \mathcal{G}_m} \langle \mathcal{P}_{f_c f_a}^{\text{gen}} \otimes \mathcal{F}_{c,b}[\mathcal{X}_m^n] + \mathcal{P}_{f_c f_b}^{\text{gen}} \otimes \mathcal{F}_{a,c} \rangle$$

where c spans the same set as a and b . Note that the Mellin convolution always acts on the momentum associated to the parton c , i.e. p_1 in the first term and p_2 in the second term. Moreover, we define the **generalized final-state anomalous dimensions** as:

$$\Gamma_{i,q} := \Gamma_{i,q \rightarrow qq} + \Gamma_{i,q \rightarrow gq} \quad \Gamma_{i,g} := \Gamma_{i,g \rightarrow gg} + 2n_f \Gamma_{i,g \rightarrow \bar{q}q} \quad (3.34)$$

Finally, we can then write the integrated hard-collinear counterterms in a compact way:

$$\begin{aligned} & \sum_{n \in \mathcal{G}_{m+1}(g)} \sum_{i \in \mathcal{H}_{m,0}^n(\mathbf{m}_g)} \langle \bar{\mathcal{S}}_m \mathcal{C}_{im} \Delta^m \mathcal{F}_{a,b}[\mathcal{X}_m^n(\mathbf{m}_g)] \rangle + \sum_{\rho=1}^{n_f} \sum_{n \in \mathcal{G}_{m+1}(q_\rho)} \sum_{i \in \mathcal{H}_{m,0}^n(\mathbf{m}_{q_\rho})} \langle \bar{\mathcal{S}}_m \mathcal{C}_{im} \Delta^m \mathcal{F}_{a,b}[\mathcal{X}_m^n(\mathbf{m}_{q_\rho})] \rangle \\ & + \sum_{\rho=1}^{n_f} \sum_{n \in \mathcal{G}_{m+1}(\bar{q}_\rho)} \sum_{i \in \mathcal{H}_{m,0}^n(\mathbf{m}_{\bar{q}_\rho})} \langle \bar{\mathcal{S}}_m \mathcal{C}_{im} \Delta^m \mathcal{F}_{a,b}[\mathcal{X}_m^n(\mathbf{m}_{\bar{q}_\rho})] \rangle \\ & = \frac{[\alpha_s]}{\epsilon} \sum_c \sum_{n \in \mathcal{G}_m} \langle \mathcal{P}_{f_c f_a}^{\text{gen}} \otimes \mathcal{F}_{c,b}[\mathcal{X}_m^n] + \mathcal{P}_{f_c f_b}^{\text{gen}} \otimes \mathcal{F}_{a,c} \rangle + [\alpha_s] \sum_{n \in \mathcal{G}_m} \langle I_C(\epsilon) \mathcal{F}_{a,b}[\mathcal{X}_m^n] \rangle \end{aligned}$$

where we used Eq. 3.34 to combine generalized process-dependent final-state anomalous dimensions into generalized final-state anomalous dimensions, which can be summed with initial-state ones to define the **integrated collinear operator**:

$$I_C(\epsilon) := \sum_{i \in \mathcal{H}_{m,0}^n} \frac{\Gamma_{i,f_i}}{\epsilon} \quad (3.35)$$

This operator is $I_C(\epsilon) \sim \epsilon^{-1}$, as opposed to $I_S(\epsilon) \sim \epsilon^{-2}$. The $I_S(\epsilon)$ and $I_C(\epsilon)$ operators allow us to rewrite the expression of the real contribution in Eq. 3.10 as:

$$\begin{aligned} 2\hat{s} d\hat{\sigma}_{a,b}^{\text{R}} &= 2\hat{s} d\hat{\sigma}_{a,b}^{\text{NLO,reg}} + [\alpha_s] \sum_{n \in \mathcal{G}_m} \langle [I_S(\epsilon) + I_C(\epsilon)] \mathcal{F}_{a,b}[\mathcal{X}_m^n] \rangle + \\ & + \frac{[\alpha_s]}{\epsilon} \sum_c \sum_{n \in \mathcal{G}_m} \langle \mathcal{P}_{f_c f_a}^{\text{gen}} \otimes \mathcal{F}_{c,b}[\mathcal{X}_m^n] + \mathcal{P}_{f_c f_b}^{\text{gen}} \otimes \mathcal{F}_{a,c}[\mathcal{X}_m^n] \rangle \quad (3.36) \end{aligned}$$

where the finite reminder is (recall Eq. 3.11):

$$d\hat{\sigma}_{a,b}^{\text{NLO,reg}} \equiv \sum_{n \in \mathcal{G}_{m+1}} \left\langle \mathcal{O}_{\text{NLO}}^m \Delta^m \left[\mathcal{F}_{a,b}[\mathcal{X}_m^n(\mathbf{m}_g)] + \sum_{\rho=1}^{n_f} (\mathcal{F}_{a,b}[\mathcal{X}_m^n(\mathbf{m}_{q_\rho})] + \mathcal{F}_{a,b}[\mathcal{X}_m^n(\mathbf{m}_{\bar{q}_\rho})]) \right] \right\rangle \quad (3.37)$$

§3.2 Virtual corrections and PDF renormalization

The counterterms resulting from the extraction of IR singularities from real corrections, which have an explicit expression in Eq. 3.36, are not ϵ -finite: their poles must cancel against those resulting from virtual corrections and from the collinear renormalization of PDFs.

§3.2.1 Virtual corrections

An expression with clear ϵ -poles can be derived from Eq. 3.4 by expressing the one-loop amplitude in Eq. 2.24 using Catani's formula:

$$|\mathcal{A}_m^{(1)}(\mu^2; \{p\})\rangle = I_1(\epsilon) |\mathcal{A}_m^{(0)}(\mu^2; \{p\})\rangle + |\mathcal{A}_m^{\text{fin}}(\mu^2; \{p\})\rangle \quad (3.38)$$

where $\mathcal{A}_m^{\text{fin}}$ is the finite part of the one-loop amplitude, and the operator $I_1(\epsilon)$ reads [27]:

$$I_1(\epsilon) = \frac{1}{2} \frac{e^{\gamma_E \epsilon}}{\Gamma(1-\epsilon)} \sum_{i \neq j \in \mathcal{X}_{m,0}^n} \frac{\mathbf{T}_i \cdot \mathbf{T}_j}{\mathbf{T}_i^2} \left(\frac{\mu^2 e^{-i\lambda_{ij}\pi}}{2p_i \cdot p_j} \right)^\epsilon \mathcal{V}_i(\epsilon) \quad (3.39)$$

with $\lambda_{ij} = +1$ if i and j are both incoming/outgoing and $\lambda_{ij} = 0$ otherwise. The singular function $\mathcal{V}_i(\epsilon)$ depends only on the parton's flavour:

$$\mathcal{V}_i(\epsilon) \equiv \frac{\mathbf{T}_i^2}{\epsilon^2} + \frac{\gamma_i}{\epsilon} \quad (3.40)$$

The I_1 operator extracts the ϵ -poles from Eq. 3.4 as:

$$2\Re \langle \mathcal{A}_m^{(0)} | \mathcal{A}_m^{(1)} \rangle = \frac{e^{\gamma_E \epsilon}}{\Gamma(1-\epsilon)} I_V(\epsilon) |\mathcal{A}_m^{(0)}|^2 + 2\Re \langle \mathcal{A}_m^{(0)} | \mathcal{A}_m^{\text{fin}} \rangle$$

where the **virtual operator** is defined as:

$$I_V(\epsilon) \equiv \frac{\Gamma(1-\epsilon)}{e^{\gamma_E \epsilon}} \left(I_1(\epsilon) + I_1^\dagger(\epsilon) \right) = \sum_{i \neq j \in \mathcal{X}_{m,0}^n} \frac{\mathbf{T}_i \cdot \mathbf{T}_j}{\mathbf{T}_i^2} \left(\frac{\mu^2}{2p_i \cdot p_j} \right)^\epsilon \cos(\lambda_{ij}\pi\epsilon) \mathcal{V}_i(\epsilon) \quad (3.41)$$

Then, making the finite reminder of the one-loop amplitude explicit, Eq. 3.4 can be rewritten as:

$$2\hat{s} d\hat{\sigma}_{a,b}^V = [\alpha_s] \sum_{n \in \mathcal{G}_m} \langle I_V(\epsilon) \mathcal{F}_{a,b}[\mathcal{X}_m^n] \rangle + \sum_{n \in \mathcal{G}_m} \langle \mathcal{F}_{a,b}^{\text{fin}}[\mathcal{X}_m^n] \rangle \quad (3.42)$$

§3.2.2 Collinear renormalization

As already stated, the partonic cross-section is a non-observable quantity and it is linked to the observable hadronic cross-section by the factorization theorem Eq. 1.3. With this theorem, non-perturbative effects are encapsulated in PDFs, which are divergent: hence, they need to be renormalized.

In the $\overline{\text{MS}}$ scheme, PDFs are renormalized multiplicatively introducing a divergent operator \mathcal{Z} which acts like (see Chapter 8 of [45]):

$$f_i^{(h)}(\xi) = \sum_j \int_{[0,1]^2} dy_1 dy_2 \mathcal{Z}_{i,j}(y_1) \bar{f}_j^{(h)}(y_2) \delta(\xi - y_1 y_2) \quad (3.43)$$

where $f_i^{(h)}$ are the bare PDFs and $\bar{f}_j^{(h)}$ are the renormalized PDFs. A convenient expression of the renormalization operator is found in [46]:

$$\mathcal{Z}_{i,j}(z) = \delta_{i,j} \delta(1-z) + \frac{\alpha_s(\mu_R^2)}{2\pi} \frac{1}{\epsilon} \hat{P}_{f_j f_i}^{(0)}(z) + o(\alpha_s^2) \quad (3.44)$$

Inserting this expression into Eq. 1.3, it is clear that the multiplicative approach is equivalent to adding NLO counterterms to the LO cross-section. For simplicity, consider the deep inelastic scattering of an electron on a hadron:

$$\begin{aligned} d\sigma_h(P) &= \sum_a \int_0^1 \frac{d\xi}{\xi} f_a^{(h)}(\xi) d\hat{\sigma}_a(\xi P) = \sum_{a,c} \int_{[0,1]^3} \frac{d\xi}{\xi} dy_1 dy_2 \mathcal{Z}_{a,c}(y_1) \bar{f}_c^{(h)}(y_2) \delta(\xi - y_1 y_2) d\hat{\sigma}_a(\xi P) \\ &= \sum_{a,c} \int_{[0,1]^2} \frac{dy_1}{y_1} \frac{dy_2}{y_2} \left[\delta_{a,c} \delta(1 - y_1) + \frac{\alpha_s(\mu_R^2)}{2\pi} \frac{1}{\epsilon} \hat{P}_{fcfa}^{(0)}(y_1) + o(\alpha_s^2) \right] \bar{f}_c^{(h)}(y_2) d\hat{\sigma}_a(y_1 y_2 P) \\ &= \sum_a \int_0^1 \frac{d\xi}{\xi} \left[\bar{f}_a^{(h)}(\xi) d\hat{\sigma}_a(\xi P) + \frac{\alpha_s(\mu_R^2)}{2\pi} \frac{1}{\epsilon} \sum_c \int_0^1 \frac{dz}{z} \hat{P}_{fcfa}^{(0)}(z) \bar{f}_c^{(h)}(\xi) d\hat{\sigma}_a(z\xi P) + o(\alpha_s^2) \right] \end{aligned}$$

The first term is the factorized cross-section with renormalized PDFs, while the second term is the NLO renormalization counterterm. It is trivial to generalize to the case of hadron-hadron scattering, obtaining Eq. 3.5.

§3.3 Integrated counterterms

As all IR poles have been extracted from Eq. 3.3-3.5, we can now show their explicit cancellation. Throughout this section we make use of the following ϵ -expansions:

$$\frac{e^{\gamma_E \epsilon}}{\Gamma(1 - \epsilon)} = 1 - \frac{\pi^2}{12} \epsilon^2 + o(\epsilon^3) \quad \frac{\Gamma^2(1 - \epsilon)}{\Gamma(1 - 2\epsilon)} = 1 - \frac{\pi^2}{6} \epsilon^2 + o(\epsilon^3) \quad (3.45)$$

As a consequence, we have:

$$[\alpha_s] = \frac{\alpha_s(\mu_R^2)}{2\pi} + o(\epsilon^2) \quad \left(\frac{2E_i}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1 - \epsilon)}{\Gamma(1 - 2\epsilon)} = 1 - 2\mathcal{L}_i \epsilon + \left(2\mathcal{L}_i^2 - \frac{\pi^2}{6} \right) \epsilon^2 + o(\epsilon^3) \quad (3.46)$$

with $k \in \mathbb{R}^+$ and \mathcal{L}_i defined in Eq. B.1.

§3.3.1 Generalized splitting functions

First, consider the third term in Eq. 3.36. It is trivial to see how these poles are cancelled by the counterterms in Eq. 3.5, e.g. consider the terms associated to the parton a :

$$\begin{aligned} \frac{[\alpha_s]}{\epsilon} \sum_c \sum_{n \in \mathcal{G}_m} \langle \mathcal{P}_{fcfa}^{\text{gen}} \otimes \mathcal{F}_{c,b}[\mathcal{X}_m^n] \rangle + \frac{\alpha_s(\mu_R^2)}{2\pi} \frac{1}{\epsilon} \sum_c \int_0^1 \frac{dz}{z} \hat{P}_{fcfa}^{(0)}(z) d\hat{\sigma}_{c,b}^{(0)}(zp_1, p_2) \\ = \frac{\alpha_s(\mu_R^2)}{2\pi} \frac{1}{\epsilon} \sum_c \sum_{n \in \mathcal{G}_m} \left\langle \int_0^1 \frac{dz}{z} \left[\mathcal{P}_{fcfa}^{\text{gen}}(z) + \hat{P}_{fcfa}^{(0)}(z) \right] \mathcal{F}_{c,b}[\mathcal{X}_m^n](zp_1) \right\rangle + o(\epsilon) \\ = \frac{\alpha_s(\mu_R^2)}{2\pi} \frac{1}{\epsilon} \sum_c \sum_{n \in \mathcal{G}_m} \left\langle \int_0^1 \frac{dz}{z} \left[\epsilon \mathcal{P}_{fcfa}^{\text{fin}}(z) + 2\epsilon \mathcal{L}_a \hat{P}_{fcfa}^{(0)}(z) + o(\epsilon^2) \right] \mathcal{F}_{c,b}[\mathcal{X}_m^n](zp_1) \right\rangle + o(\epsilon) \\ = \frac{\alpha_s(\mu_R^2)}{2\pi} \sum_c \sum_{n \in \mathcal{G}_m} \left\langle \int_0^1 \frac{dz}{z} \left[\mathcal{P}_{fcfa}^{\text{fin}}(z) + 2\mathcal{L}_a \hat{P}_{fcfa}^{(0)}(z) \right] \mathcal{F}_{c,b}[\mathcal{X}_m^n](zp_1) \right\rangle + o(\epsilon) \end{aligned}$$

where we used Eq. 3.28. This expression is manifestly ϵ -finite.

§3.3.2 Integrated operators

We now turn our attention to the sum of the integrated operators, and define the operator $I_T \equiv I_S + I_C + I_V$. To show that $I_T(\epsilon)$ does not contain any pole, first consider the sum $I_S + I_V$:

$$I_S(\epsilon) + I_V(\epsilon) = \sum_{i \neq j \in \mathcal{X}_{m,0}^n} (\mathbf{T}_i \cdot \mathbf{T}_j) \left[-\frac{1}{\epsilon^2} \left(\frac{2\mathcal{E}}{\mu} \right)^{-2\epsilon} K_{ij} + \left(\frac{\mu^2}{2p_i \cdot p_j} \right)^\epsilon \cos(\lambda_{ij}\pi\epsilon) \frac{\mathcal{V}_i(\epsilon)}{\mathbf{T}_i^2} \right]$$

where we defined:

$$K_{ij} \equiv \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \eta_{ij} {}_2F_1(1, 1, 1-\epsilon, 1-\eta_{ij}) = 1 - \epsilon \log \eta_{ij} + o(\epsilon^2) \quad (3.47)$$

Then, writing $2p_i \cdot p_j = 4E_i E_j \eta_{ij}$:

$$I_S(\epsilon) + I_V(\epsilon) = \sum_{i \neq j \in \mathcal{X}_{m,0}^n} (\mathbf{T}_i \cdot \mathbf{T}_j) \frac{1}{\epsilon^2} \left(\frac{2\mathcal{E}}{\mu} \right)^{-2\epsilon} \left[-K_{ij} + \left(\frac{\mathcal{E}^2}{E_i E_j} \right)^\epsilon \eta_{ij}^{-\epsilon} \cos(\lambda_{ij}\pi\epsilon) \left(1 + \epsilon \frac{\gamma_i}{\mathbf{T}_i^2} \right) \right]$$

Expanding the last factor in square brackets, we get:

$$-1 + \epsilon \log \eta_{ij} + 1 + \epsilon (L_i + L_j) - \epsilon \log \eta_{ij} + \epsilon \frac{\gamma_i}{\mathbf{T}_i^2} + o(\epsilon^2) = \epsilon \left(L_i + L_j + \frac{\gamma_i}{\mathbf{T}_i^2} \right) + o(\epsilon^2)$$

Hence:

$$I_S(\epsilon) + I_V(\epsilon) = \frac{1}{\epsilon} \sum_{i \neq j \in \mathcal{X}_{m,0}^n} (\mathbf{T}_i \cdot \mathbf{T}_j) \left(L_i + L_j + \frac{\gamma_i}{\mathbf{T}_i^2} \right) + o(\epsilon^0) \quad (3.48)$$

We can make use of the colour algebra to further simplify this sum. In this case $\mathcal{X}_{m,0}^n \equiv \mathcal{H}_{m,0}^n$, so:

$$\sum_{i \neq j \in \mathcal{X}_{m,0}^n} (\mathbf{T}_i \cdot \mathbf{T}_j) \mathcal{O}_i = \sum_{i \in \mathcal{H}_{m,0}^n} \mathbf{T}_i \cdot \left[\sum_{j \in \mathcal{H}_{m,0}^n} \mathbf{T}_j \mathcal{O}_j - \mathbf{T}_i \mathcal{O}_i \right] = - \sum_{i \in \mathcal{H}_{m,0}^n} \mathbf{T}_i^2 \mathcal{O}_i \quad (3.49)$$

where the sum over $j \in \mathcal{H}_{m,0}^n$ was removed due to the colour-conservation condition Eq. 2.31. Then, the above sum can be rewritten as:

$$I_S(\epsilon) + I_V(\epsilon) = -\frac{1}{\epsilon} \sum_{i \in \mathcal{H}_{m,0}^n} (\gamma_i + 2\mathbf{T}_i^2 L_i) + o(\epsilon^0) \quad (3.50)$$

On the other hand, the integrated collinear operator is:

$$I_C(\epsilon) = \frac{1}{\epsilon} \sum_{i \in \mathcal{H}_{m,0}^n} \Gamma_{i,fi} = \frac{1}{\epsilon} (\Gamma_{a,fa} + \Gamma_{b,fb}) + \frac{1}{\epsilon} \sum_{i \in \mathcal{H}_m^n} \Gamma_{i,fi}$$

The expansion of initial-state generalized anomalous dimensions is clear from Eq. 3.27:

$$\Gamma_{a,fa} = \gamma_a + 2\mathbf{T}_a^2 L_a + o(\epsilon) \quad (3.51)$$

Final-state generalized anomalous dimensions are less immediate to expand. Indeed, explicitly from Eq. 3.21, 3.34:

$$\begin{aligned}
\Gamma_{i,q} &= - \left(\frac{2E_i}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \left[\int_0^1 dz \bar{S}_z \frac{P_{qq}(z) + P_{qg}(z)}{z^{-1+2\epsilon}(1-z)^{2\epsilon}} - C_F \frac{1 - e^{-2\epsilon L_i}}{\epsilon} \right] \\
&= - \left(\frac{2E_i}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \left[\int_0^1 dz \bar{S}_z \frac{P_{qq}(z)}{z^{2\epsilon}(1-z)^{2\epsilon}} - C_F \frac{1 - e^{-2\epsilon L_i}}{\epsilon} \right] \\
&= - \int_0^1 dz \bar{S}_z \frac{P_{qq}(z)}{z^{2\epsilon}(1-z)^{2\epsilon}} + C_F^2 \frac{1 - e^{-2\epsilon L_i}}{\epsilon} + o(\epsilon) = \gamma_q + 2C_F L_i + o(\epsilon)
\end{aligned}$$

where we used the property $P_{qg}(z) = P_{qg}(1-z)$, and:

$$\begin{aligned}
\Gamma_{i,g} &= - \left(\frac{2E_i}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \left[\int_0^1 dz \bar{S}_z \frac{P_{gg}(z) + 2n_f P_{gq}(z)}{z^{-1+2\epsilon}(1-z)^{2\epsilon}} - C_A \frac{1 - e^{-2\epsilon L_i}}{\epsilon} \right] \\
&= \frac{11}{6} C_A - \frac{2}{3} T_R n_f + 2C_A L_i + o(\epsilon) = \gamma_g + 2C_A L_i + o(\epsilon)
\end{aligned}$$

Therefore, they have the same expression as Eq. 3.51. Hence, we can write the integrated collinear operator as:

$$I_C(\epsilon) = \frac{1}{\epsilon} \sum_{i \in \mathcal{H}_{m,0}^n} (\gamma_i + 2\mathbf{T}_i^2 L_i) + o(\epsilon^0) \quad (3.52)$$

The poles in Eq. 3.50, 3.52 manifestly cancel, leaving then $I_T(\epsilon) \sim o(\epsilon^0)$. Denoting $I_T(\epsilon) \equiv I_T^{(0)} + o(\epsilon)$, the NLO correction in Eq. 1.7 is written in an ϵ -finite form as:

$$\begin{aligned}
d\hat{\sigma}_{a,b}^{(1)} &= d\hat{\sigma}_{a,b}^{\text{NLO,reg}} + \frac{1}{2\hat{s}} \sum_{n \in \mathcal{G}_m} \left[\frac{\alpha_s(\mu_R^2)}{2\pi} \left\langle I_T^{(0)} \mathcal{F}_{a,b}[\mathcal{X}_m^n] \right\rangle + \left\langle \mathcal{F}_{a,b}^{\text{fin}}[\mathcal{X}_m^n] \right\rangle \right] \\
&\quad + \frac{\alpha_s(\mu_R^2)}{2\pi} \sum_c \left[P_{fc,fa}^{\text{gen}} \otimes d\hat{\sigma}_{c,b}^{(0)} + P_{fc,fb}^{\text{gen}} \otimes d\hat{\sigma}_{a,c}^{(0)} \right] \quad (3.53)
\end{aligned}$$

where we defined:

$$P_{fcfa}^{\text{gen}}(z) \equiv \mathcal{P}_{fcfa}^{\text{fin}}(z) + 2\mathcal{L}_a \hat{P}_{fcfa}^{(0)}(z) \quad (3.54)$$

The various terms of Eq. 3.53 have no IR singularity, thus they can be numerically integrated. We note that this equation correctly reproduces the results in [32].

NSC SS with Massive Quarks

The NSC SS, as developed in [32–34] and described in the previous chapter, does not include massive final-state partons. In this chapter, we consider the inclusion of these massive partons: we prove the cancellation of poles in this case, i.e. with $\mathcal{X}_m^n(Q) \neq \emptyset$, $\mathcal{X}_m^n(\bar{Q}) \neq \emptyset$, and we derive the integrated counterterms at NLO in an explicit form.

The general structure of the NSC SS as illustrated in the previous chapter remains valid: in particular, massive partons do not change collinear singularities (as shown in §1.2.1), hence the $I_C(\epsilon)$ operator, as well as the Mellin convolutions of generalized splitting functions, remain unchanged. The only operators that do change are $I_S(\epsilon)$ and $I_V(\epsilon)$.

§4.1 Generalized operators

§4.1.1 Generalized soft operator

The factorization formula for amplitudes in the soft limit, given in Eq. 3.13, holds in general, thus to generalize the soft operator we only have to compute integrated eikonal factors in the case of massive partons. To do so, we choose a different parametrization of the phase space of the parton \mathbf{m}_g , with $p^\mu = (p_0, p_1, p_2, p_\perp)$:

$$d^{d-1}p = dp_1 dp_2 d^{d-3}p_\perp = dp_1 dp_2 dp_\perp p_\perp^{d-4} d\Omega_{d-4} \quad (4.1)$$

Then, we perform a transformation to polar coordinates $(p_1, p_2, p_\perp) \mapsto (p_0, \vartheta, \varphi)$ such that:

$$p_1 = p_0 \cos \vartheta \quad p_2 = p_0 \sin \vartheta \cos \varphi \quad p_\perp = p_0 \sin \vartheta \sin \varphi \quad (4.2)$$

The condition $p_\perp \in \mathbb{R}_0^+$ implies $\varphi \in [0, \pi]$, while $\vartheta \in [0, \pi]$ by definition. As $p_0 = E$ for gluons, the phase-space measure becomes:

$$[dp] \equiv \frac{d^{3-2\epsilon}p}{(2\pi)^{3-2\epsilon}2E} \theta(\mathcal{E} - E) = \frac{d\Omega_{-2\epsilon}}{2(2\pi)^{3-2\epsilon}} dE E^{1-2\epsilon} \theta(\mathcal{E} - E) d\cos \vartheta d\varphi (\sin \vartheta \sin \varphi)^{-2\epsilon} \quad (4.3)$$

Now we can perform the integration of the eikonal factor $\mathcal{S}_{i,j}$:

$$\begin{aligned} \int [dp_{\mathbf{m}}] \mathcal{S}_{i,j} &= \int_{\mathbb{S}^{-2\epsilon}} \frac{d\Omega_{-2\epsilon}}{2(2\pi)^{3-2\epsilon}} \int_0^{\mathcal{E}} \frac{dE_{\mathbf{m}}}{E_{\mathbf{m}}^{1-2\epsilon}} \int_{-1}^1 d\cos \vartheta \int_0^\pi d\varphi (\sin \vartheta \sin \varphi)^{-2\epsilon} \frac{\hat{p}_i \cdot \hat{p}_j}{(\hat{p}_i \cdot \hat{p}_{\mathbf{m}})(\hat{p}_j \cdot \hat{p}_{\mathbf{m}})} \\ &= -\frac{2^{1-2\epsilon} \pi^{-\epsilon}}{2(2\pi)^{3-2\epsilon}} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \frac{\mathcal{E}^{-2\epsilon}}{2\epsilon} \pi I_{i,j} = -\frac{1}{\epsilon} \frac{\pi^\epsilon}{16\pi^2} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \mathcal{E}^{-2\epsilon} I_{i,j}(\epsilon) \end{aligned}$$

where we used the following identity:

$$\int_{\mathbb{S}^{d-1}} d\Omega_{d-1} = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} = 2^d \pi^{\frac{d}{2}-\frac{1}{2}} \frac{\Gamma(\frac{d+1}{2})}{\Gamma(d)} \quad (4.4)$$

and defined the generalized angular integral:

$$I_{i,j} \equiv \frac{1}{\pi} \int_{-1}^1 d\cos\vartheta \int_0^\pi d\varphi (\sin\vartheta \sin\varphi)^{-2\epsilon} \frac{\hat{p}_i \cdot \hat{p}_j}{(\hat{p}_i \cdot \hat{p}_m)(\hat{p}_j \cdot \hat{p}_m)} \quad (4.5)$$

with $\hat{p}_k : p_k^\mu = E_k \hat{p}_k^\mu \forall k \in \mathcal{X}_{m,0}^n \cup \{\mathbf{m}\}$. It only remains to evaluate $I_{i,j}(\epsilon)$ in the three possible cases for i and j : massive–massive, massive–massless (note that $I_{i,j}(\epsilon) = I_{j,i}(\epsilon)$) and massless–massless. The former two cases are known in the literature, e.g. Appendix D.2 of [47], and are reported in §B.3.

To determine $I_{i,j}(\epsilon)$ in the massless–massless case, we just have to compare the expression for the integrated eikonal factor computed in §3.1.2 with the above expression:

$$\frac{\mathcal{E}^{-2\epsilon}}{\epsilon^2} \eta_{ij} \frac{\pi^\epsilon}{8\pi^2} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} {}_2F_1(1, 1, 1-\epsilon, 1-\eta_{ij}) = -\frac{1}{\epsilon} \frac{\pi^\epsilon}{16\pi^2} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \mathcal{E}^{-2\epsilon} I_{i,j}(\epsilon)$$

We can now make use of the following expansion:

$$\eta {}_2F_1(1, 1, 1-\epsilon, 1-\eta) = 1 - \epsilon \log|\eta| - \frac{1}{2} (\log^2|\eta| + 2 \text{Li}_2(1-\eta)) \epsilon^2 + o(\epsilon^3) \quad (4.6)$$

Hence, with the same notation from §B.3, we can write:

$$I_{i,j}^{(-1)} = -2 \quad I_{i,j}^{(0)} = 2 \log|\eta_{ij}| \quad I_{i,j}^{(1)} = \log^2|\eta_{ij}| + 2 \text{Li}_2(1-\eta_{ij}) \quad (4.7)$$

Inserting everything in Eq. 3.13, we get:

$$\sum_{n \in \mathcal{G}_{m+1}(g)} \langle S_m \Delta^m \mathcal{F}_{a,b}[\mathcal{X}_m^n(\mathbf{m}_g)] \rangle = [\alpha_s] \langle I_S(\epsilon) \mathcal{F}_{a,b}[\mathcal{X}_m^n] \rangle \quad (4.8)$$

with the generalized soft operator:

$$I_S(\epsilon) := \frac{1}{2\epsilon} \left(\frac{2\mathcal{E}}{\mu} \right)^{-2\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \sum_{i,j \in \mathcal{X}_{m,0}^n} (\mathbf{T}_i \cdot \mathbf{T}_j) I_{i,j}(\epsilon) \quad (4.9)$$

§4.1.2 Generalized virtual operator

Recall that the IR poles of the one-loop amplitude for a process involving massless partons can be expressed in terms of Catani's operator $I_1(\epsilon)$ (see Eq. 3.39): a generalization of this operator in the case of massive partons is given in [39]. However, as already stated, this operator is written in a charge-unrenormalized way, hence we have to perform the renormalization of the results of this reference. In the following, we will denote charge-unrenormalized amplitudes as \mathcal{M} and charge-renormalized amplitudes as \mathcal{A} .

Loop corrections to the LO amplitude can be written as:

$$\mathcal{M}_m = \left(\frac{\alpha_{s,b}}{4\pi\mu^{2\epsilon}} \right)^q \left[\mathcal{M}_m^{(0)} + \frac{\alpha_{s,b}}{4\pi\mu^{2\epsilon}} \mathcal{M}^{(1)} + o(\alpha_{s,b}^2) \right] \quad (4.10)$$

where $q \in \frac{1}{2}\mathbb{N}_0$. Note that this equation fixes the normalization of the charge-unrenormalized amplitudes. Then, the IR poles can be written as:

$$\mathcal{M}_m^{(1)} = I_{\text{CDR}}(\epsilon) \mathcal{M}_m^{(0)} + \mathcal{M}_m^{\text{fin}} \quad (4.11)$$

where $\mathcal{M}_m^{\text{fin}}$ is the non-singular part of the one-loop amplitude and Catani's operator¹ is defined as (in CDR):

$$I_{\text{CDR}}(\epsilon) := \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \left[q \frac{\beta_0}{\epsilon} + I_1(\epsilon) \right] \quad (4.12)$$

$$I_1(\epsilon) := \sum_{i \neq j \in \mathcal{X}_{m,0}^n} (\mathbf{T}_i \cdot \mathbf{T}_j) \left(\frac{\mu^2}{|s_{ij}|} \right)^\epsilon \left[\mathcal{V}_{i,j}(\epsilon) + \frac{1}{v_{ij}} \left(\frac{i\pi}{\epsilon} - \frac{\pi^2}{2} \right) \theta(s_{ij}) \right] - \sum_{i \in \mathcal{X}_{m,0}^n} \Gamma_i(\epsilon) \quad (4.13)$$

with v_{ij} defined in Eq. B.21, $s_{ij} \equiv 2p_i \cdot p_j$ and the singular functions defined in §B.4. Inserting Eq. 4.11-4.13 into Eq. 4.10:

$$\begin{aligned} \mathcal{M}_m = \left(\frac{\alpha_{s,b}}{4\pi\mu^{2\epsilon}} \right)^q & \left[\left(1 + \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \frac{\alpha_{s,b}}{4\pi\mu^{2\epsilon}} q \frac{\beta_0}{\epsilon} \right) \mathcal{M}_m^{(0)} \right. \\ & \left. + \frac{\alpha_{s,b}}{4\pi\mu^{2\epsilon}} \left(\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} I_1(\epsilon) \mathcal{M}_m^{(0)} + \mathcal{M}_m^{\text{fin}} \right) + o(\alpha_{s,b}^2) \right] \end{aligned}$$

Renormalization in the $\overline{\text{MS}}$ scheme is performed via Eq. 2.18. Substituting this to $\alpha_{s,b}$ yields:

$$\begin{aligned} \mathcal{M}_m &= \left(\frac{\alpha_s}{4\pi S_\epsilon} \right)^q \left[1 - \frac{1}{2} \frac{\alpha_s}{2\pi} \frac{\beta_0}{\epsilon} + o(\alpha_s^2) \right]^q \times \\ &\quad \times \left[\left(1 + \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \frac{\alpha_s}{4\pi S_\epsilon} q \frac{\beta_0}{\epsilon} \right) \mathcal{M}_m^{(0)} + \frac{\alpha_s}{4\pi S_\epsilon} \left(\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} I_1(\epsilon) \mathcal{M}_m^{(0)} + \mathcal{M}_m^{\text{fin}} \right) + o(\alpha_{s,b}^2) \right] \\ &= \left(\frac{\alpha_s}{4\pi S_\epsilon} \right)^q \left[\left(1 - \frac{\alpha_s}{4\pi} q \frac{\beta_0}{\epsilon} \left(1 - \frac{e^{\gamma_E \epsilon}}{\Gamma(1-\epsilon)} \right) \right) \mathcal{M}_m^{(0)} + \right. \\ &\quad \left. + \frac{\alpha_s}{4\pi} \frac{e^{\gamma_E \epsilon}}{\Gamma(1-\epsilon)} I_1(\epsilon) \mathcal{M}_m^{(0)} + \frac{\alpha_s}{4\pi S_\epsilon} \mathcal{M}_m^{\text{fin}} + o(\alpha_s^2) \right] \\ &= \left(\frac{\alpha_s}{4\pi S_\epsilon} \right)^q \left[\mathcal{M}_m^{(0)} + \frac{\alpha_s}{4\pi} \frac{e^{\gamma_E \epsilon}}{\Gamma(1-\epsilon)} I_1(\epsilon) \mathcal{M}_m^{(0)} + \frac{\alpha_s}{4\pi S_\epsilon} \mathcal{M}_m^{\text{fin}} + o(\epsilon) + o(\alpha_s^2) \right] \end{aligned}$$

Hence, we can write the renormalized relation:

$$\mathcal{A}_m = \left(\frac{\alpha_s}{2\pi} \right)^q \left[\mathcal{A}_m^{(0)} + \frac{\alpha_s}{2\pi} \mathcal{A}_m^{(1)} + o(\alpha_s^2) \right] \quad (4.14)$$

with:

$$\mathcal{A}_m \equiv \mathcal{M}_m \quad \mathcal{A}_m^{(0)} \equiv (2S_\epsilon)^{-q} \mathcal{M}_m^{(0)} \quad \mathcal{A}_m^{\text{fin}} \equiv (2S_\epsilon)^{-(1+q)} \mathcal{M}_m^{\text{fin}} \quad (4.15)$$

$$\mathcal{A}_m^{(1)} = \frac{1}{2} \frac{e^{\gamma_E \epsilon}}{\Gamma(1-\epsilon)} I_1(\epsilon) \mathcal{A}_m^{(0)} + \mathcal{A}_m^{\text{fin}} \quad (4.16)$$

It is then clear, from Eq. 3.4, that:

$$2\hat{s} \, d\hat{\sigma}_{a,b}^V = [\alpha_s] \sum_{n \in \mathcal{G}_m} \langle I_V(\epsilon) \mathcal{F}_{a,b}[\mathcal{X}_m^n] \rangle + \sum_{n \in \mathcal{G}_m} \langle \mathcal{F}_{a,b}^{\text{fin}}[\mathcal{X}_m^n] \rangle \quad (4.17)$$

where the generalized virtual operator is:

$$I_V(\epsilon) := \Re I_1(\epsilon) = \sum_{i \neq j \in \mathcal{X}_{m,0}^n} (\mathbf{T}_i \cdot \mathbf{T}_j) \left(\frac{\mu^2}{|s_{ij}|} \right)^\epsilon \left[\mathcal{V}_{i,j}(\epsilon) - \frac{1}{v_{ij}} \frac{\pi^2}{2} \theta(s_{ij}) \right] - \sum_{i \in \mathcal{X}_{m,0}^n} \Gamma_i(\epsilon) \quad (4.18)$$

§4.2 Integrated counterterms

The generalized expressions for the soft and virtual operators with massive partons derived in the previous section can be combined with the unchanged collinear operator to form $I_T \equiv I_S + I_V + I_C$. It now only remains to show that $I_T(\epsilon)$ is ϵ -finite and find its $o(\epsilon^0)$ term $I_T^{(0)}$. These constitute the main results of this work. First, setting $I_{S+V} \equiv I_S + I_V$, we show the cancellation of colour-correlated terms between the soft and virtual operators (Eq. 4.9, 4.18):

$$\begin{aligned} I_{S+V}(\epsilon) &= \sum_{i \neq j \in \mathcal{X}_{m,0}^n} (\mathbf{T}_i \cdot \mathbf{T}_j) \left[\frac{1}{2\epsilon} \left(1 - 2\mathcal{L}_m \epsilon + \left(2\mathcal{L}_m^2 - \frac{\pi^2}{6} \right) \epsilon^2 + o(\epsilon^3) \right) I_{i,j}(\epsilon) + \right. \\ &\quad \left. + \left(1 + \epsilon \log \frac{\mu^2}{|s_{ij}|} + \frac{1}{2} \epsilon^2 \log^2 \frac{\mu^2}{|s_{ij}|} + o(\epsilon^3) \right) \left(\mathcal{V}_{i,j}(\epsilon) - \frac{1}{v_{ij}} \frac{\pi^2}{2} \theta(s_{ij}) \right) \right] + \\ &\quad + \sum_{i \in \mathcal{X}_{m,0}^n} \left[\mathbf{T}_i^2 \frac{1}{2\epsilon} (1 - 2\mathcal{L}_m \epsilon + o(\epsilon^2)) I_{i,i}(\epsilon) - \Gamma_i(\epsilon) \right] \\ &= \sum_{i \neq j \in \mathcal{X}_{m,0}^n} (\mathbf{T}_i \cdot \mathbf{T}_j) \left[\left(\frac{1}{2\epsilon} - \mathcal{L}_m + \left(\mathcal{L}_m^2 - \frac{\pi^2}{12} \right) \epsilon \right) \left(\frac{I_{i,j}^{(-1)}}{\epsilon} + I_{i,j}^{(0)} + \epsilon I_{i,j}^{(1)} \right) + o(\epsilon) \right. \\ &\quad \left. + \left(1 + \epsilon \log \frac{\mu^2}{|s_{ij}|} + \frac{1}{2} \epsilon^2 \log^2 \frac{\mu^2}{|s_{ij}|} \right) \left(\frac{\mathcal{V}_{i,j}^{(-2)}}{\epsilon^2} + \frac{\mathcal{V}_{i,j}^{(-1)}}{\epsilon} + \mathcal{V}_{i,j}^{(0)} - \frac{1}{v_{ij}} \frac{\pi^2}{2} \theta(s_{ij}) \right) + o(\epsilon) \right] + \\ &\quad + \sum_{i \in \mathcal{X}_{m,m}^n} \left[\mathbf{T}_i^2 \left(\frac{1}{2\epsilon} - \mathcal{L}_m \right) \left(I_{i,i}^{(0)} + \epsilon I_{i,i}^{(1)} \right) - \Gamma_i(\epsilon) \right] - \sum_{i \in \mathcal{H}_{m,0}^n} \Gamma_i(\epsilon) + o(\epsilon^2) \end{aligned}$$

where we set $\mathcal{X}_{m,m}^n \equiv \mathcal{X}_m^n(Q) \cup \mathcal{X}_m^n(\bar{Q})$. To better understand the calculation, we consider each summation separately. The colour-correlated term can be written as:

$$\begin{aligned} &\underbrace{\frac{1}{\epsilon^2} \left(\frac{1}{2} I_{i,j}^{(-1)} + \mathcal{V}_{i,j}^{(-2)} \right) + \frac{1}{\epsilon} \left(\frac{1}{2} I_{i,j}^{(0)} + \mathcal{V}_{i,j}^{(-1)} - \mathcal{L}_m I_{i,j}^{(-1)} + \mathcal{V}_{i,j}^{(-2)} \log \frac{\mu^2}{|s_{ij}|} \right)}_{\chi_{i,j}} + o(\epsilon) + \\ &\underbrace{+ \frac{1}{2} I_{i,j}^{(1)} - \mathcal{L}_m I_{i,j}^{(0)} + \left(\mathcal{L}_m^2 - \frac{\pi^2}{12} \right) I_{i,j}^{(-1)} + \mathcal{V}_{i,j}^{(0)} - \frac{1}{v_{ij}} \frac{\pi^2}{2} \theta(s_{ij}) + \mathcal{V}_{i,j}^{(-1)} \log \frac{\mu^2}{|s_{ij}|} + \frac{1}{2} \mathcal{V}_{i,j}^{(-2)} \log^2 \frac{\mu^2}{|s_{ij}|}}_{\mathfrak{X}_{i,j}} \end{aligned}$$

where $\chi_{i,j}$ contains the ϵ -poles and $\mathfrak{X}_{i,j}$ is ϵ -finite (more precisely, $\mathfrak{X}_{i,j} \sim o(\epsilon^0)$). The explicit expression of $\chi_{i,j}$ is derived in §A.4: in particular, it has ϵ -poles only if i and/or j are massless

¹Note that the notation is slightly different from [39].

partons, while it vanishes otherwise. The sum on $i \in \mathcal{X}_{m,m}^n$ instead reads, using Eq. B.25,B.31:

$$\sum_{i \in \mathcal{X}_{m,m}^n} \left[C_F \left(\frac{1}{\epsilon} - \frac{1}{\kappa_i} \log \frac{1 - \kappa_i}{1 + \kappa_i} - 2\mathcal{L}_m \right) - C_F \left(\frac{1}{\epsilon} + \frac{1}{2} \log \frac{m_{Q_i}^2}{\mu^2} - 2 \right) \right] \equiv \sum_{i \in \mathcal{X}_{m,m}^n} \mathfrak{A}_i$$

which is manifestly ϵ -finite. Finally, from Eq. B.29-B.30, the last summation expands as:

$$\sum_{i \in \mathcal{H}_{m,0}^n} \Gamma_i(\epsilon) = \sum_{i \in \mathcal{H}_{m,0}^n} \left(\frac{1}{\epsilon} \gamma_i + \mathfrak{U}_i \right)$$

where we have defined the $o(\epsilon^0)$ term as:

$$\mathfrak{U}_g \equiv -\frac{2}{3} T_R \sum_{\rho=1}^{n_F} \log \frac{m_{Q_\rho}^2}{\mu^2} \quad \mathfrak{U}_q \equiv 0 \quad (4.19)$$

We can put all the above expressions together, yielding:

$$I_{S+V}(\epsilon) = \sum_{i \neq j \in \mathcal{X}_{m,0}^n} (\mathbf{T}_i \cdot \mathbf{T}_j) [\chi_{i,j} + \mathfrak{K}_{i,j}] + \sum_{i \in \mathcal{X}_{m,m}^n} \mathfrak{A}_i - \sum_{i \in \mathcal{H}_{m,0}^n} \left(\frac{1}{\epsilon} \gamma_i + \mathfrak{U}_i \right) \quad (4.20)$$

Note that this equation appears to have colour-correlated poles, contained in $\chi_{i,j}$. We now prove that these in fact cancel, as was the case for massless amplitudes. Recalling that $\mathcal{X}_{m,0}^n$ is the set of all partons, $\mathcal{H}_{m,0}^n$ the set of all massless partons and $\mathcal{X}_{m,m}^n$ the set of all final-state massive partons, writing $\mathcal{X}_{m,0}^n \times \mathcal{X}_{m,0}^n = (\mathcal{H}_{m,0}^n \cup \mathcal{X}_{m,m}^n) \times (\mathcal{H}_{m,0}^n \cup \mathcal{X}_{m,m}^n)$, we find:

$$\sum_{i \neq j \in \mathcal{X}_{m,0}^n} (\mathbf{T}_i \cdot \mathbf{T}_j) \chi_{i,j} = \left[\sum_{i \neq j \in \mathcal{H}_{m,0}^n} + \sum_{i \neq j \in \mathcal{X}_{m,m}^n} + \sum_{i \in \mathcal{H}_{m,0}^n, j \in \mathcal{X}_{m,m}^n} + \sum_{i \in \mathcal{X}_{m,m}^n, j \in \mathcal{H}_{m,0}^n} \right] (\mathbf{T}_i \cdot \mathbf{T}_j) \chi_{i,j}$$

From §A.4, $\chi_{i,j}$ contributes with a L_i/ϵ term for each massless index, hence:

$$\begin{aligned} \sum_{i \neq j \in \mathcal{X}_{m,0}^n} (\mathbf{T}_i \cdot \mathbf{T}_j) \chi_{i,j} &= \sum_{i \neq j \in \mathcal{H}_{m,0}^n} (\mathbf{T}_i \cdot \mathbf{T}_j) \frac{L_i + L_j}{\epsilon} + \\ &\quad + \sum_{i \in \mathcal{H}_{m,0}^n, j \in \mathcal{X}_{m,m}^n} (\mathbf{T}_i \cdot \mathbf{T}_j) \frac{L_i}{\epsilon} + \sum_{i \in \mathcal{X}_{m,m}^n, j \in \mathcal{H}_{m,0}^n} (\mathbf{T}_i \cdot \mathbf{T}_j) \frac{L_j}{\epsilon} \\ &= \sum_{i,j \in \mathcal{H}_{m,0}^n} (\mathbf{T}_i \cdot \mathbf{T}_j) \frac{L_i + L_j}{\epsilon} - \sum_{i \in \mathcal{H}_{m,0}^n} \mathbf{T}_i^2 \frac{2L_i}{\epsilon} + \\ &\quad + \sum_{i \in \mathcal{H}_{m,0}^n, j \in \mathcal{X}_{m,m}^n} (\mathbf{T}_i \cdot \mathbf{T}_j) \frac{L_i}{\epsilon} + \sum_{i \in \mathcal{X}_{m,m}^n, j \in \mathcal{H}_{m,0}^n} (\mathbf{T}_i \cdot \mathbf{T}_j) \frac{L_j}{\epsilon} \\ &= \sum_{i \in \mathcal{H}_{m,0}^n} \frac{L_i}{\epsilon} \mathbf{T}_i \cdot \sum_{j \in \mathcal{X}_{m,0}^n} \mathbf{T}_j + \sum_{i \in \mathcal{X}_{m,0}^n} \mathbf{T}_i \cdot \sum_{j \in \mathcal{H}_{m,0}^n} \mathbf{T}_j \frac{L_j}{\epsilon} - \sum_{i \in \mathcal{H}_{m,0}^n} \mathbf{T}_i^2 \frac{2L_i}{\epsilon} \\ &= - \sum_{i \in \mathcal{H}_{m,0}^n} \mathbf{T}_i^2 \frac{2L_i}{\epsilon} \end{aligned}$$

where we used the colour-conservation condition Eq. 2.31. Finally, we find an expression for the sum of soft and virtual operators without colour-correlated poles:

$$I_{S+V}(\epsilon) = -\frac{1}{\epsilon} \sum_{i \in \mathcal{H}_{m,0}^n} (\gamma_i + 2\mathbf{T}_i^2 L_i) + I_{S+V}^{(0)} + o(\epsilon) \quad (4.21)$$

where the $o(\epsilon^0)$ term reads:

$$I_{S+V}^{(0)} \equiv \sum_{i \neq j \in \mathcal{X}_{m,0}^n} (\mathbf{T}_i \cdot \mathbf{T}_j) \mathbf{K}_{i,j} + \sum_{i \in \mathcal{X}_{m,m}^n} \mathbf{A}_i - \sum_{i \in \mathcal{H}_{m,0}^n} \mathbf{U}_i \quad (4.22)$$

The poles in Eq. 4.21 clearly cancel those in the collinear operator Eq. 3.52, as it remains unchanged when including massive final-state partons. Thus, we confirm that the cancellation of poles holds in the extension of the NSC SS including massive final states in NLO corrections. We now only need to determine its Laurent series at $o(\epsilon^0)$. First, consider the generalized initial-state anomalous dimension Eq. 3.27:

$$\Gamma_{a,f_a} = \gamma_a + 2\mathbf{T}_a^2 L_a - \epsilon \left[2\mathcal{L}_a (\gamma_a + 2\mathbf{T}_a^2 L_a) + 2\mathbf{T}_a^2 L_a^2 \right] + o(\epsilon^2) \quad (4.23)$$

Then, the generalized final-state anomalous dimension Eq. 3.21, 3.34:

$$\Gamma_{i,q} = \gamma_q + 2C_F L_i - \epsilon \left[2\mathcal{L}_i (\gamma_q + 2C_F L_i) + 2C_F L_i^2 - \frac{C_F}{6} (39 - 4\pi^2) \right] + o(\epsilon^2) \quad (4.24)$$

$$\Gamma_{i,g} = \gamma_g + 2C_A L_i - \epsilon \left[2\mathcal{L}_i (\gamma_g + 2C_A L_i) + 2C_A L_i^2 - \frac{C_A}{9} (67 - 6\pi^2) + \frac{23}{9} T_R n_f \right] + o(\epsilon^2) \quad (4.25)$$

Therefore, the Laurent series of the collinear operator is:

$$I_C(\epsilon) = \frac{1}{\epsilon} \sum_{i \in \mathcal{H}_{m,0}^n} (\gamma_i + 2\mathbf{T}_i^2 L_i) + I_C^{(0)} + o(\epsilon) \quad (4.26)$$

where the $o(\epsilon^0)$ term reads:

$$\begin{aligned} I_C^{(0)} = & - \sum_{i \in \mathcal{H}_{m,0}^n} \left[2\mathcal{L}_i (\gamma_i + 2\mathbf{T}_i^2 L_i) + 2\mathbf{T}_i^2 L_i^2 \right] + \\ & + (N_q + N_{\bar{q}}) \frac{C_F}{6} (39 - 4\pi^2) + N_g \left[\frac{C_A}{9} (67 - 6\pi^2) - \frac{23}{9} T_R n_f \right] \end{aligned} \quad (4.27)$$

where $N_g \equiv |\mathcal{X}_m^n(g)|$ is the number of final-state gluons, $N_q \equiv |\mathcal{X}_m^n(q)|$ the number of final-state massless quarks and $N_{\bar{q}} \equiv |\mathcal{X}_m^n(\bar{q})|$ the number of final-state massless antiquarks.

We see that, when including massive final-state partons, the general structure of the NLO correction Eq. 3.53 remains unchanged. Aside from the amplitudes themselves, the only term that does change, indeed, is the $I_T^{(0)}$ term, whose expression is obtained combining Eq. 4.22, 4.27.

Conclusions

The generalized soft and virtual operators Eq. 4.9, 4.18, as well as the $\sim o(\epsilon^0)$ integrated counterterms Eq. 4.22, constitute the main results of this work. Indeed, they correctly account for the presence of generic massive partons in the final state: as expected, the integrated counterterms contain terms of the kind $\sim \log m^2/\mu^2$ (in particular, Π_i and \mathcal{D}_i have this explicit logarithmic dependence on quark masses, while $\mathcal{K}_{i,j}$ contains these terms in the massive–massive and massive–massless $\mathcal{V}_{i,j}$ coefficients).

These results pave the way for the inclusion of massive final-state partons in NLO calculations performed within the NSC subtraction framework. The explicit form of the counterterms is reported in §B.5: Π_i and \mathcal{D}_i are expressed in a form which can be readily integrated on the LO-like multi-parton final-state phase space through Monte Carlo techniques, while $\mathcal{K}_{i,j}$ still appears in a colour-correlated sum in the final $\sim o(\epsilon^0)$ Laurent coefficient. A further improvement of these results would then be the (partial) expression of the colour-correlated sum in Eq. 4.22 as a function of Casimir operators (i.e. $\sim \mathbf{T}_i^2$ terms), in order to facilitate its numerical integration: indeed, integrating colour-correlated terms requires both more effort in writing code to handle colour correlation and more computing time to run it.

Moreover, this work lays the foundations for a future generalization of the NSC SS with massive partons at NNLO: as explained in [32], many of the singularities encountered in NNLO calculations are captured by the I_S , I_C and I_V operators or by their iterations, hence having already generalized their expressions to include massive final-state partons should significantly reduce the work needed for a generalization of the NNLO calculation including massive partons. In particular, since final-state massive partons do not affect collinear limits or the collinear renormalization of PDFs, and given that double-virtual corrections at $o(\epsilon^{-2})$ can be written in terms of $I_V(\epsilon)$, $I_V(2\epsilon)$ and $[I_1(\epsilon), I_1^\dagger(\epsilon)]$ (see [27]), the only difficulty left is generalizing the various soft limits at NNLO to include both massless and massive partons, as we did in §4.1.1. Indeed, work in this direction has already begun: see e.g. [48, 49].

Finally, we note that we only considered final-state massive partons in this work. The reason for this limitation is that the inclusion of initial-state massive partons is currently poorly understood: indeed, parton distribution functions are generally defined for massless partons, but their definition does not strictly require this condition, hence the factorization theorem Eq. 1.3 is still valid in presence of initial-state massive partons. However, the collinear renormalization of PDFs changes, as the masses of initial-state partons regulate the corresponding collinear divergences: fixed-order diagrams that would be collinearly divergent for $m \rightarrow 0$ instead produce a logarithmic dependence in the form of powers $\log^n Q^2/m^2$, with Q the energy scale of the hard process. We see that, for $Q \gg m$, these logarithms can spoil the convergence of the perturbative expansion Eq. 1.7, as each term is now $\sim \alpha_s^n \log^n Q^2/m^2$: the best strategy in this case would be the resummation of massive logarithms, which is an area of active research.

Appendices

Appendix A

Mathematical reference

§A.1 Phase-space parametrization

In dimensional regularization with $d = 4 - 2\epsilon$, we define the measure on the phase space of a parton i to be:

$$[dp_i] \equiv \frac{d^{d-1}p_i}{(2\pi)^{d-1}2E_i} \theta(\mathcal{E} - E_i) \quad (\text{A.1})$$

Note that \mathcal{E} is an upper bound on the energies of individual partons: it is an arbitrary parameter to be taken sufficiently large as to be greater or equal to the maximal energy that a final-state parton can reach.

This measure can be cast in a more useful form introducing a suitable parametrization of the phase space: in particular, given that $\mathbb{R}^n - \{\mathbf{0}\} \cong \mathbb{R}^+ \times \mathbb{S}^{n-1}$, it is convenient to introduce hyperspherical coordinates on the \mathbb{S}^{d-2} component of the phase space. In general, the **hyperspherical measure** on \mathbb{S}^n is recursively defined as:

$$d\Omega_n = \sin^{n-1} \varphi \, d\varphi \, d\Omega_{n-1} \quad (\text{A.2})$$

Using Eq. A.2 (with $\sin \varphi \, d\varphi = d \cos \varphi$), we can express the measure $d^{d-1}p_i$ as:

$$d^{d-1}p_i = |\mathbf{p}_i|^{d-2} \, d|\mathbf{p}_i| \, \sin^{d-4} \varphi \, d \cos \varphi \, d\Omega_{d-3} \quad (\text{A.3})$$

As we are only interested in integrations on phase spaces of real unresolved partons, which can only be massless, we can use the on-shell condition $p_i^2 = 0$ to express $|\mathbf{p}_i| = E_i$, so that the phase-space measure becomes:

$$[dp_i] = \theta(\mathcal{E} - E_i) E_i^{d-3} dE_i \sin^{d-4} \varphi \, d \cos \varphi \, \frac{d\Omega_{d-3}}{2(2\pi)^{d-1}} \quad (\text{A.4})$$

with $E_i \in \mathbb{R}^+$ and $\varphi \in [0, \pi]$.

§A.1.1 Multi-particle phase space

When considering scattering processes, in general the final state is a multi-particle state, hence the measure on the final-state phase space must account for energy conservation too.

Given a $2 \rightarrow m$ scattering process with well-defined initial momenta p_a and p_b , then the differential cross-section is (see Chapter 4 of [16]):

$$d\sigma = \frac{1}{2E_a 2E_b |\mathbf{v}_a - \mathbf{v}_b|} \prod_{k=1}^m \int \frac{d^3p_k}{(2\pi)^3 2E_k} |\mathcal{A}(ab \rightarrow \mathcal{H})|^2 (2\pi)^4 \delta^{(4)}(p_a + p_b - \sum_{i=1}^m p_i) \quad (\text{A.5})$$

where $\mathcal{A}(ab \rightarrow \mathcal{H})$ is the amplitude of the scattering process and $\mathbf{v}_k \equiv \frac{\mathbf{p}_k}{E_k}$ is the velocity of the k^{th} particle.

As we are only interested in massless initial-state partons, in the center-of-mass (CM) frame $p_{a,b} = (E, \pm \mathbf{p})$, hence it is trivial to see that the flux factor in Eq. A.5 is just $2\hat{s} := 2(p_a + p_b)^2$. The differential cross-section can then be rewritten as:

$$d\sigma = \frac{1}{2\hat{s}} \int d\Phi_m (2\pi)^4 \delta^{(4)}(p_a + p_b - \sum_{i=1}^m p_i) |\mathcal{M}(ab \rightarrow \mathcal{H})|^2 \quad (\text{A.6})$$

where the **invariant m -body phase space measure** is defined as:

$$d\Phi_m \equiv \prod_{k=1}^m [dp_k] \quad (\text{A.7})$$

§A.2 Partitions of unity

To define a partition of unity such as Eq. 3.9, define $\mathcal{H}^i \equiv \mathcal{H}_{m+1}^n - \{i\}$ and introduce the function:

$$d^i \equiv \prod_{k \in \mathcal{H}^i} p_{k,\perp} \prod_{l < m \in \mathcal{H}^i} \rho_{lm} \quad (\text{A.8})$$

where $p_{k,\perp}$ is the transverse momentum of the parton k . Then, the partition is found as:

$$\Delta^i := \frac{d^i}{\sum_{j \in \mathcal{H}_{m+1}^n} d^j} \quad (\text{A.9})$$

These clearly provide a \mathcal{H}_{m+1}^n -partition of unity. To prove its properties, note that:

$$S_i d^j = \lim_{E_i \rightarrow 0} \prod_{k \in \mathcal{H}^i} p_{k,\perp} \prod_{l < m \in \mathcal{H}^i} \rho_{lm} = \begin{cases} 0 & i \neq j \\ d_j & i = j \end{cases}$$

Thus trivially $S_i \Delta^j = \delta_i^j$. The collinear limit is slightly more complex:

$$C_{ij} d^k = \lim_{\rho_{ij} \rightarrow 0} \prod_{s \in \mathcal{H}^i} p_{s,\perp} \prod_{l < m \in \mathcal{H}^i} \rho_{lm} = \begin{cases} 0 & i, j \neq k \\ d^k & i = k \neq j \end{cases}$$

The latter case has two possibilities: either $j \in \{a, b\}$ or $j \in \mathcal{H}^i$. Then, clearly $C_{ia} d^i = C_{ib} d^i = 1$, while if $j \in \mathcal{H}^i$:

$$C_{ij} \Delta^i = \frac{d^i}{d^i + d^j} = \left[1 + \frac{d^j}{d^i} \right]^{-1}$$

With explicit calculation:

$$\frac{d^j}{d^i} = \frac{p_{i,\perp}}{p_{j,\perp}} \frac{\rho_{1,j} \cdots \rho_{i-1,j} \rho_{i+1,j} \cdots \rho_{j-1,j} \rho_{j,j+1} \cdots \rho_{j,m+1}}{\rho_{1,i} \cdots \rho_{i-1,i} \rho_{i,i+1} \cdots \rho_{i,j-1} \rho_{i,j+1} \cdots \rho_{i,m+1}} = \frac{p_{i,\perp}}{p_{j,\perp}} = \frac{E_i}{E_j}$$

where we have used the fact that $\rho_{ik} \rightarrow \rho_{jk} \forall k \neq i, j$ as $\rho_{ij} \rightarrow 0$. This completes the proof.

To construct the angular partition of unity in Eq. 3.12, we set $g_{kl} \equiv \rho_{kl}^{-1}$ and define the angular factors:

$$\omega^{mi} := \frac{g_{im}}{\sum_{j \in \mathcal{H}_{m,0}^n(\mathbf{m})} g_{jm}} \quad (\text{A.10})$$

These clearly provide a $\mathcal{H}_{m,0}^n(\mathbf{m})$ -partition of unity. Moreover:

$$C_{jm} \omega^{mi} = \lim_{\rho_{jm} \rightarrow 0} \frac{g_{im}}{\sum_{k \in \mathcal{H}_{m,0}^n(\mathbf{m})} g_{km}} = \left[\lim_{\rho_{jm} \rightarrow 0} \sum_{k \in \mathcal{H}_{m,0}^n(\mathbf{m})} \frac{\rho_{im}}{\rho_{km}} \right]^{-1} = \begin{cases} 1^{-1} & j = i \\ \infty^{-1} & j \neq i \end{cases} = \delta_j^i$$

where we made an abuse of notation. This completes the proof.

§A.3 Quadratic Casimir operators of $\text{SU}(n_c)$

To prove Eq. 2.16, first consider the fundamental representation \mathbf{n} of $\text{SU}(n_c)$. Then, contracting Eq. 2.17 with δ^{ab} (with $a, b = 1, \dots, n_c^2 - 1$, as they label the basis of $\mathfrak{su}(n_c)$):

$$C_2(\mathbf{n})n_c = T_R(n_c^2 - 1)$$

To compute the Casimir operator for the adjoint representation \mathbf{g} , consider the decomposition of the direct product of two representations:

$$\mathbf{r}_1 \otimes \mathbf{r}_2 = \bigoplus_i \mathbf{r}_i$$

In this representation $T_{\mathbf{r}_1 \otimes \mathbf{r}_2}^a = T_{\mathbf{r}_1}^a \otimes \text{id}_{\mathbf{r}_2} + \text{id}_{\mathbf{r}_1} \otimes T_{\mathbf{r}_2}^a$, and it acts on tensor objects Ξ_{pq} whose first index transforms according to \mathbf{r}_1 and the second index according to \mathbf{r}_2 . Recalling that $\text{tr } T^a = 0$:

$$\begin{aligned} \text{tr}(T_{\mathbf{r}_1 \otimes \mathbf{r}_2}^a)^2 &= \text{tr}((T_{\mathbf{r}_1}^a)^2 \otimes \text{id}_{\mathbf{r}_2} + 2T_{\mathbf{r}_1}^a \otimes T_{\mathbf{r}_2}^a + \text{id}_{\mathbf{r}_1} \otimes (T_{\mathbf{r}_2}^a)^2) \\ &= \text{tr}(C_2(\mathbf{r}_1) \text{id}_{\mathbf{r}_1} \otimes \text{id}_{\mathbf{r}_2}) + \text{tr}(C_2(\mathbf{r}_2) \text{id}_{\mathbf{r}_1} \otimes \text{id}_{\mathbf{r}_2}) = (C_2(\mathbf{r}_1) + C_2(\mathbf{r}_2))n_{\mathbf{r}_1}n_{\mathbf{r}_2} \end{aligned}$$

However, by the decomposition above:

$$\text{tr}(T_{\mathbf{r}_1 \otimes \mathbf{r}_2}^a)^2 = \sum_i C_2(\mathbf{r}_i)n_{\mathbf{r}_i}$$

Consider $\mathbf{n} \otimes \mathbf{n}^*$, where \mathbf{n}^* is the complex conjugate of the fundamental representation (for complex representations, \mathbf{r} and \mathbf{r}^* are generally inequivalent representations): then Ξ_{pq} contains a term proportional to the invariant δ_{pq} , while the other $n_c^2 - 1$ independent components transform as a general $n_c \times n_c$ traceless tensor, i.e. under the adjoint representation of $\text{SU}(n_c)$ (as of Eq. 2.12-2.13), thus $\mathbf{n} \otimes \mathbf{n}^* = \mathbf{1} \oplus \mathbf{g}$ and the above identity becomes:

$$(C_2(\mathbf{1}) + C_2(\mathbf{g}))(n_c^2 - 1) = (C_2(\mathbf{n}) + C_2(\mathbf{n}^*))n_c^2$$

Using $C_2(\mathbf{1}) = 0$ (as all generators are trivially zero) and $C_2(\mathbf{n}^*) = C_2(\mathbf{n})$:

$$C_2(\mathbf{g})(n_c^2 - 1) = 2T_R \frac{n_c^2 - 1}{n_c} n_c^2$$

which completes the proof.

§A.4 Pole coefficients

In this section, we derive an explicit expression for the $\chi_{i,j}$ coefficients defined in §4.2. Their expression only depends on whether i and j are massive–massive, massive–massless or massless–massless.

Massive–massive The required coefficients for $I_{i,j}$ and $\mathcal{V}_{i,j}$ in the massive–massive case are listed in Eq. B.22,B.26, so:

$$\chi_{i,j} = \frac{1}{\epsilon} \left(\frac{1}{2} \frac{1}{v_{ij}} \log \frac{1+v_{ij}}{1-v_{ij}} + \frac{1}{2} \frac{1}{v_{ij}} \log \frac{1-v_{ij}}{1+v_{ij}} \right)$$

Hence, trivially:

$$\chi_{i,j} = 0 \quad (\text{A.11})$$

Massive–massless In the case of i massive and j massless, we use Eq. B.24,B.27:

$$\begin{aligned} \chi_{i,j} &= \frac{1}{\epsilon} \left(\frac{1}{2} \log \frac{(\hat{p}_i \cdot \hat{p}_j)^2}{\hat{p}_i \cdot \hat{p}_i} + \frac{1}{2} \log \frac{m_i^2}{|s_{ij}|} + \log \frac{2\mathcal{E}}{\mu} + \frac{1}{2} \log \frac{\mu^2}{|s_{ij}|} \right) \\ &= \frac{1}{2\epsilon} \left(\log \frac{4E_i^2 \eta_{ij}^2}{m_i^2} + \log \frac{m_i^2}{4E_i E_j |\eta_{ij}|} + \log \frac{4\mathcal{E}^2}{\mu^2} + \log \frac{\mu^2}{4E_i E_j |\eta_{ij}|} \right) = \frac{1}{2\epsilon} \log \frac{\mathcal{E}^2}{E_j^2} \end{aligned}$$

Thus:

$$\chi_{i,j} = \frac{1}{\epsilon} L_j \quad (\text{A.12})$$

The case of i massless and j massive is trivially found with $i \leftrightarrow j$.

Massless–massless Finally, for the massless–massless case we have Eq. 4.7,B.28, i.e.:

$$\chi_{i,j} = \frac{1}{\epsilon} \left(\log |\eta_{ij}| + 2 \log \frac{2\mathcal{E}}{\mu} + \log \frac{\mu^2}{|s_{ij}|} \right) = \frac{1}{\epsilon} \left(\log |\eta_{ij}| + \log \frac{4\mathcal{E}^2}{\mu^2} + \log \frac{\mu^2}{4E_i E_j |\eta_{ij}|} \right)$$

Therefore:

$$\chi_{i,j} = \frac{1}{\epsilon} (L_i + L_j) \quad (\text{A.13})$$

Appendix B

Collection of relevant equations

In this Appendix, we provide definitions of relevant objects used in this work. In order to simplify various equations, we use a notation analogous to [34]:

$$\begin{aligned} \bar{z} &\equiv 1 - z & \mathcal{D}_n(z) &\equiv \left[\frac{\log^n(1-z)}{1-z} \right]_+ \\ L_i &\equiv \log \frac{\mathcal{E}}{E_i} & \mathcal{L}_i &\equiv \log \frac{2E_i}{\mu_R} & \mathcal{L}_m &\equiv \log \frac{2\mathcal{E}}{\mu_R} \end{aligned} \quad (\text{B.1})$$

Moreover, we adopt the following short-hands:

$$\rho_{ij} \equiv 1 - \cos \theta_{ij} \quad \eta_{ij} \equiv \frac{\rho_{ij}}{2} \quad (\text{B.2})$$

where i and j are two partons.

§B.1 Useful constants

Denoting the colour-charge operators \mathbf{T}_i , with the conventional normalization $T_R = \frac{1}{2}$ for $\text{SU}(n_c)$, the squares of these operators are the quadratic Casimir operators of the corresponding representations:

$$\mathbf{T}_q^2 = \mathbf{T}_{\bar{q}}^2 = C_F = \frac{n_c^2 - 1}{2n_c} \quad \mathbf{T}_g^2 = C_A = n_c \quad (\text{B.3})$$

The quark and gluon anomalous dimensions are:

$$\gamma_q = \frac{3}{2}C_F \quad \gamma_g = \frac{11}{6}C_A - \frac{2}{3}T_R n_q \quad (\text{B.4})$$

where n_q is the number of active flavours.

The strong coupling is renormalized in the $\overline{\text{MS}}$ scheme, so that the bare and running couplings are related by:

$$\alpha_{s,b} S_\epsilon = \alpha_s(\mu_R^2) \mu_R^{2\epsilon} \left[1 - \frac{\alpha_s(\mu_R^2)}{2\pi} \frac{\beta_0}{\epsilon} + o(\alpha_s^2) \right] \quad (\text{B.5})$$

where $S_\epsilon \equiv (4\pi)^\epsilon e^{-\gamma_E \epsilon}$ and:

$$\beta_0 = \frac{11}{6}C_A - \frac{2}{3}T_R n_q = \gamma_g \quad (\text{B.6})$$

It is convenient to define a quantity related to the running coupling:

$$[\alpha_s] \equiv \frac{\alpha_s(\mu_R^2)}{2\pi} \frac{e^{\gamma_E \epsilon}}{\Gamma(1-\epsilon)} \quad (\text{B.7})$$

§B.2 Splitting functions

Consider the final-state splitting process $[i\mathbf{m}]^* \rightarrow i(z) + \mathbf{m}(1-z)$, where i and \mathbf{m} are two partons of flavours f_i and $f_{\mathbf{m}}$ and $[i\mathbf{m}]$ is the corresponding clustered parton of flavour $f_{[i\mathbf{m}]}$. Recall that, given the interaction vertices determined by the QCD Lagrangian Eq. 2.11 (see §2.1.3), a gluon clustered with any type of parton preserves the latter's flavours, while a quark clustered with an antiquark gives a gluon.

The energy fraction carried by the parton i is defined as $z \equiv 1 - E_{\mathbf{m}}/E_{[i\mathbf{m}]}$. As a consequence, the parton \mathbf{m} carries an energy fraction $1 - z$. Denoting the spin-averaged final-state splitting functions as $P_{f_{[i\mathbf{m}]}f_i}(z)$, they read:

$$P_{qq}(z) = C_F \left[\frac{1+z^2}{1-z} - \epsilon(1-z) \right] \quad (\text{B.8})$$

$$P_{qg}(z) = C_F \left[\frac{1-(1-z)^2}{z} - \epsilon z \right] \equiv P_{qg}(1-z) \quad (\text{B.9})$$

$$P_{gq}(z) = T_R \left[1 - \frac{2z(1-z)}{1-\epsilon} \right] \quad (\text{B.10})$$

$$P_{gg}(z) = 2C_A \left[\frac{z}{1-z} + \frac{1-z}{z} + z(1-z) \right] \quad (\text{B.11})$$

Now, consider instead the initial-state splitting process $i \rightarrow [i\mathbf{m}]^* + \mathbf{m}$, where i and \mathbf{m} are respectively an ingoing and outgoing parton, while the clustered parton $[i\mathbf{m}]^*$ enters the hard scattering process. In this case, we define the z variable as $z \equiv 1 - E_{\mathbf{m}}/E_i$. The spin- and colour-averaged initial-state splitting functions, denoted as $P_{f_{[i\mathbf{m}]}f_i,i}(z)$, are:

$$P_{qq,i} = -zP_{qq}(1/z) \equiv P_{qq}(z) \quad (\text{B.12})$$

$$P_{qg,i} = \left[\frac{2n_c}{2(1-\epsilon)(n_c^2-1)} \right] zP_{qg}(1/z) \equiv P_{qg}(z) \quad (\text{B.13})$$

$$P_{gq,i} = \left[\frac{2(1-\epsilon)(n_c^2-1)}{2n_c} \right] zP_{gq}(1/z) \equiv P_{gq}(z) \quad (\text{B.14})$$

$$P_{gg,i} = -zP_{gg}(1/z) \equiv P_{gg}(z) \quad (\text{B.15})$$

Finally, the LO Altarelli-Parisi splitting kernels are:

$$\hat{P}_{qq}^{(0)}(z) = C_F \left[2\mathcal{D}_0(z) - (1+z) + \frac{3}{2}\delta(1-z) \right] \quad (\text{B.16})$$

$$\hat{P}_{qg}^{(0)}(z) = T_R \left[(1-z)^2 + z^2 \right] \quad (\text{B.17})$$

$$\hat{P}_{gq}^{(0)}(z) = C_F \left[\frac{1+(1-z)^2}{z} \right] \quad (\text{B.18})$$

$$\hat{P}_{gg}^{(0)}(z) = 2C_A \left[\mathcal{D}_0(z) + z(1-z) + \frac{1}{z} - 2 \right] + \beta_0\delta(1-z) \quad (\text{B.19})$$

All these splitting functions and kernels can be found in [23].

§B.3 Massive angular integrals

As in [47], we can expand the $I_{i,j}$ integral defined in Eq. 4.5 in its Laurent series in ϵ :

$$I_{i,j} = \sum_{k=-1}^1 \epsilon^k I_{i,j}^{(k)} + o(\epsilon^2) \quad (\text{B.20})$$

The expression of the coefficients $I_{i,j}^{(k)}$ depends on the nature of the partons i and j .

Massive–massive If i and j are both massive, then $p_i^2 = m_i^2$ and $p_j^2 = m_j^2$. Set the notation:

$$v_{ij} \equiv \sqrt{1 - \frac{m_i^2 m_j^2}{(p_i \cdot p_j)^2}} \quad \mathbf{q}_i \equiv \frac{\mathbf{p}_i}{E_i} \quad \mathbf{q}_j \equiv \frac{\mathbf{p}_j}{E_j} \quad (\text{B.21})$$

Moreover, define the shorthands:

$$\begin{aligned} A^2 &= (\mathbf{q}_i - \mathbf{q}_j)^2 & X_1 &= \mathbf{q}_i^2 - (\mathbf{q}_i \cdot \mathbf{q}_j) \\ B^2 &= \mathbf{q}_i^2 \mathbf{q}_j^2 - (\mathbf{q}_i \cdot \mathbf{q}_j)^2 & X_2 &= \mathbf{q}_j^2 - (\mathbf{q}_i \cdot \mathbf{q}_j) \end{aligned}$$

and the arguments:

$$\gamma_{\pm} = A \pm \sqrt{A^2 - B^2} \quad z_k = \sqrt{X_k^2 + B^2} - X_k$$

used in the function:

$$K(z) = -2 \text{Li}_2 \left(\frac{2\gamma_-(\gamma_+ - z)}{(\gamma_+ - \gamma_-)(\gamma_- + z)} \right) - 2 \text{Li}_2 \left(\frac{-2\gamma_+(\gamma_- + z)}{(\gamma_+ - \gamma_-)(\gamma_+ - z)} \right) - \frac{1}{2} \log^2 \frac{(z - \gamma_-)(\gamma_+ - z)}{(z + \gamma_-)(\gamma_+ + z)}$$

With this notation, the coefficients $I_{i,j}^{(k)}$ in the massive–massive case read:

$$I_{i,j}^{(-1)} = 0 \quad I_{i,j}^{(0)} = \frac{1}{v_{ij}} \log \frac{1 + v_{ij}}{1 - v_{ij}} \quad I_{i,j}^{(1)} = \frac{1 - (\mathbf{q}_i \cdot \mathbf{q}_j)}{\sqrt{A^2 - B^2}} [K(z_2) - K(z_1)] \quad (\text{B.22})$$

Massive–massless As $I_{i,j} = I_{j,i}$, consider WLOG i massive and j massless, i.e. $p_i^2 = m_i^2$ and $p_j^2 = 0$. In this case, we define:

$$\kappa_i \equiv \sqrt{1 - \frac{m_i^2}{E_i^2}} \quad \hat{p}_i \equiv \frac{p_i}{E_i} \quad \hat{p}_j \equiv \frac{p_j}{E_j} \quad (\text{B.23})$$

Then, the coefficients $I_{i,j}^{(k)}$ in the massive–massless case read:

$$\begin{aligned} I_{i,j}^{(-1)} &= -1 & I_{i,j}^{(0)} &= \log \frac{(\hat{q}_i \cdot \hat{q}_j)^2}{\hat{q}_i \cdot \hat{q}_j} \\ I_{i,j}^{(1)} &= -2 \left[\frac{1}{4} \log^2 \frac{1 - \kappa_i}{1 + \kappa_i} + \log \frac{\hat{q}_i \cdot \hat{q}_j}{1 + \kappa_i} \log \frac{\hat{q}_i \cdot \hat{q}_j}{1 - \kappa_i} + \text{Li}_2 \left(1 - \frac{\hat{q}_i \cdot \hat{q}_j}{1 + \kappa_i} \right) + \text{Li}_2 \left(1 - \frac{\hat{q}_i \cdot \hat{q}_j}{1 - \kappa_i} \right) \right] \end{aligned} \quad (\text{B.24})$$

Self-correlated massive In the special case $i = j$, if i is massless then $I_{i,i} = 0$ trivially from the definition of the eikonal factor, while if i is massive then the coefficients $I_{i,i}^{(k)}$ read:

$$I_{i,i}^{(-1)} = 0 \quad I_{i,i}^{(0)} = 2 \quad I_{i,i}^{(1)} = -\frac{2}{\kappa_i} \log \frac{1 - \kappa_i}{1 + \kappa_i} \quad (\text{B.25})$$

where κ_i is defined in Eq. B.23.

§B.4 Generalized Catani's functions

In the generalization of Catani's operator (Eq. 4.13), two divergent functions appear. Consider first the $\mathcal{V}_{i,j}(\epsilon)$ functions. If i and j are both massive, then:

$$\mathcal{V}_{i,j}(\epsilon) = \frac{1}{2\epsilon} \frac{1}{v_{ij}} \log \frac{1 - v_{ij}}{1 + v_{ij}} - \frac{1}{4} \left(\log^2 \frac{m_i^2}{|s_{ij}|} + \log^2 \frac{m_j^2}{|s_{ij}|} \right) - \frac{\pi^2}{6} \quad (\text{B.26})$$

If i is massive and j is massless, instead:

$$\mathcal{V}_{i,j}(\epsilon) = \frac{1}{2\epsilon^2} + \frac{1}{2\epsilon} \log \frac{m_i^2}{|s_{ij}|} - \frac{1}{4} \log^2 \frac{m_i^2}{|s_{ij}|} - \frac{\pi^2}{12} \quad (\text{B.27})$$

Finally, if i and j are both massless, $\mathcal{V}_{i,j}(\epsilon)$ reduces to:

$$\mathcal{V}_{i,j}(\epsilon) = \frac{1}{\epsilon^2} \quad (\text{B.28})$$

Now, consider the $\Gamma_i(\epsilon)$ functions. They only depend on whether i is a gluon, a massless quark or a massive quark, and the corresponding expressions are:

$$\Gamma_g(\epsilon) = \frac{1}{\epsilon} \gamma_g - \frac{2}{3} T_R \sum_{\rho=1}^{n_F} \log \frac{m_{Q_\rho}^2}{\mu^2} \quad (\text{B.29})$$

$$\Gamma_q = \frac{1}{\epsilon} \gamma_q \quad (\text{B.30})$$

$$\Gamma_Q = C_F \left[\frac{1}{\epsilon} + \frac{1}{2} \log \frac{m_Q^2}{\mu^2} - 2 \right] \quad (\text{B.31})$$

§B.5 Integrated finite remainders

We report the various finite remainders present in the $o(\epsilon^0)$ term of the Laurent series of $I_{S+V}(\epsilon)$ (Eq. 4.22). The colour-correlated remainders, defined for $i \neq j$, are:

$$\begin{aligned} \mathbb{K}_{i,j} = & \frac{1}{2} I_{i,j}^{(1)} - \mathcal{L}_m I_{i,j}^{(0)} + \left(\mathcal{L}_m^2 - \frac{\pi^2}{12} \right) I_{i,j}^{(-1)} + \mathcal{V}_{i,j}^{(0)} + \\ & - \frac{1}{v_{ij}} \frac{\pi^2}{2} \theta(s_{ij}) + \mathcal{V}_{i,j}^{(-1)} \log \frac{\mu^2}{|s_{ij}|} + \frac{1}{2} \mathcal{V}_{i,j}^{(-2)} \log^2 \frac{\mu^2}{|s_{ij}|} \end{aligned} \quad (\text{B.32})$$

Then, the other finite remainders are:

$$\mathfrak{D}_i = -C_F \left[\frac{1}{\kappa_i} \log \frac{1 - \kappa_i}{1 + \kappa_i} + \frac{1}{2} \log \frac{m_{Q_i}^2}{\mu^2} - 2 \right] \quad (\text{B.33})$$

which is defined for massive partons, and:

$$\mathfrak{U}_i = -\delta_{f_i, g} \frac{2}{3} T_R \sum_{\rho=1}^{n_F} \log \frac{m_{Q_\rho}^2}{\mu^2} \quad (\text{B.34})$$

which is defined for massless partons.

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